ENRICHMENT TRAINING COURSE FOR RESOURCE PERSONS IN MATHEMATICS

AT

UPPER PRIMARY LEVEL

RESOURCE MATERIAL - I

PROJECT COORDINATOR

SURJA KUMARI

DEPARTMENT OF EDUCATION IN SCIENCE AND MATHEMATICS
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING
N I E CAMPUS - SRI AUROBINDO MARG
NEW DELHI-110016

FOREWORD

The great achievements of science and technology in all its forms, which deeply influence the life of every human being, have led to a widespread recognition of importance of mathematics. Everybody knows or at least believes that without the use of mathematical tools these achievements could not have come about. As a consequence mathematics has occuped an important place in the school curriculum.

The Department of Education in Science and Mathematics at National Institute of Education(NIE), New Delhi has launched several projects for teachers and students to improve science and mathematics education in the country. One of such projects is Enrichment Training Course for Resource Persons in Mathematics for upper primary stage. The purpose of the course is to acquaint and equip the resource persons with new thrust areas of mathematics who can then train the teachers in their respective states. In this connection, a training course was organised both at national and state level for the resource persons.

The initial planning of the project was done by Dr. Surja Kumari of this Department. The lectures delivered by various teacher educators and content experts are compiled in the form of a report so that the material can be used in future training programmes. I am thankful to Prof. J.N. Kapur, Dr. N. Puri, Prof. Asha Rani Singal, Dr. S.P. Arya, Dr. Sunder Lal, Dr. Hukum Singh, Dr. B. Deokinandan and Dr. Surja Kumari who delivered the lectures and acquainted the participants with innovative methods of mathematics teaching in the classroom.. Thanks are also due to participants who showed keen

interest in this teaching-learning activity of the workshop. I appreciate the efforts of Dr. Surja Kumari for coordinating the workshop and compiling the report.

Suggestions for improvement are welcome.

(B. Ganguly)
Professor & Head of the Deptt. &
Dean(Academic)
Department of Education in Science and Mathematics
National Council of Educational Research and Training
Sri Aurobindo Marg : New Delhi - 110016.

VEDIC MATHEMATICS

MATHS

OF

SMILES

by

Dr. Narinder Puri
Reader in Civil Engineering
University of Roorkee
ROORKEE

Secretary
Vedic Maths Research Group

Lecture Notes on Vedic Mathematics
delivered at the Maths Enrichment Course of NCERT
at Delhi

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LECTURE - 1

1. INTRODUCTION

Vedic Maths system is most suitable for total development of human personality. Vedic Maths sustematically trains the mind to be alert and agile as we are always required to look for patterns. This helps in development of both halves of the brain whereas the present system trains only the left half). Further, multiple choices make Maths a playful subject.

2. REVISION

The first Vedic sutra is Ekadhiken Purven ( ). It means One more than the previous one.

Let us make friends with numbers.

Ekadhika of 0 is 1 (one)

Ekadhika of 1 is 2 (two)

Ekadhika of 2 is 3 (three)

Ekadhika of 3 is 4 (four)

Ekadhika of 4 is 5 (five)

Ekadhika of 5 is 6 (six)

Ekadhika of 6 is 7 (seven)

Ekadhika of 7 is 8 (eight)

Ekadhika of 8 is 9 (nine)

3. BEST FRIEND

We all have a best friend.

We like him very much.

We love him very much.

We play wih our best friend.

Similarly

Every number has a best friend.

Let us learn the names of the best friend.

The best friend of 1 is 9.

The best friend of One is Nine.

The best friend of 2 is 8.

The best friend of Two is Eight.

The best friend of 3 is 7.

The best friend of Three is Seven.

The best friend of 4 is 6.

The best friend of Four is Six.

And How Funny

Best friend of 5 is 5.

The best friend of Five is Five.

If Ram is best friend of Shyam. Always

Shyam is best friend of Ram .

Similarly

Best friend of 4 is 6.

Best friend of Four is Six, and

Best friend of 6 is 4.

Best friend of 7 is 3.

Best friend of Seven is Three.

Best friend of 8 is 2.
Best friend of Eight is Two.

Best friend of 9 is 1.

Best friend of Nine is One.

4. PURAK FROM TEN

Each friend is called the Purak (   ) (complement) of the other number.

So Purak of 1 is 9.
Purak of 2 is 8.
Purak of 3 is 7.
Purak of 4 is 6.
Purak of 5 is 5.
Purak of 6 is 4.
Purak of 7 is 3.
Purak of 8 is 2.
Purak of 9 is 1.

5. EKANYUNENA SUTRA

Now we shall learn another Vedic Sutra which is reverse of the first Vedic sutra. It reads Ekanyunena Purvena (   ) No. 14. It means one less than the previous one .We have already seen that as the numbers are increasing from 1 to 9 by Ekadhiken Purvena, their Purak number is decreasing by Ekanyunena Purvena.

Numbers: 1 2 3 4 5 6 7 8 9
9 8 7 6 5 4 3 2 1

6. PROPERTY OF BEST FRIENDS

We can use any digit or we can use best friend of any digit.

Whenever we use the Purak (the best friend), the best

friend comes with a cap on its head.

The Purak is written with a bar on the digit and called Rekhank.

So Purak of 1 is written as $\bar{9}$.
Purak of 2 is written as $\bar{8}$.
Purak of 3 is written as $\bar{7}$.
Purak of 4 is written as $\bar{6}$.
Purak of 5 is written as $\bar{5}$.
Purak of 6 is written as $\bar{4}$.
Purak of 7 is written as $\bar{3}$.
Purak of 8 is written as $\bar{2}$.
Purak of 9 is written as $\bar{1}$.

The best friend behaves in opposite manner. When we add bar digit - Rekhank, it is subtracted. When we subtract bar digit -Rekhank it is added.

7. PROPERTIES OF PURAK

Purak of each number is written with a bar. A Rekhank (a digit with a bar) is a negative digit. $\bar{1}$ is like one extra pencil with me which is a loan from my friend, which has to be given back. Similar is the case with $\bar{3}$, $\bar{4}$ etc.

The best friend or purak of each digit behaves in opposite manner. When we add Rekhank - bar digit it is subtracted. When we subtract bar digit it is added.

NOTE:- Very simple operations of bar digit should be slowly taught to the children step by step on the lines of

the given examples.

Examples -1

$2 + 1 = 3$

But $2 + \bar{1} = 1$

$2 - 1 = 1$

$2 - \bar{1} = 3$

$4 + \bar{3} = 1$

$\bar{3} + 4 = 1$

$5 + \bar{3} = 2$

$5 + \bar{5} = 0$

$6 + \bar{3} = 3$

$3 + 2 = 5$

$3 + \bar{2} = 1$

$4 - 3 = 1$

$4 - \bar{3} = 7$

$7 + 2 = 9$

$7 + \bar{2} = 5$

$9 + \bar{2} = 7$

$6 + 3 = 9$

$3 + \bar{3} = 0$

NOTE:- Lot of problems like this may be given till the child clearly learns the operation of bar digits. The digits used should be small so that the result is a single digit. This initial practice shall be very useful for future working with Vinculum numbers, having both positive and negative digits. We teachers may ourselves need more practice of this simple concept which leads to lot of simplification and ease in later working of Mathematics. All positive digits should be larger.

8. COMPARING NUMBERS

Each number is Ekadhika of previous number. Therefore, each number is one more than the previous one. Hence each number is greater than (>) all the numbers before it and lesser than all the number after it.

Ekadhika of 107 is 108. Therefore 108 is greater than 107 or 108 > 107.

Further, 108 is greater than all the previous numbers. Therefore

108 > 103   108 > 98

108 > 26   108 > 9

And also 107 is less than 108 or 107 < 108.

Further, 107 < 109   107 < 152

107 < 900   etc.

We know that 9 is the Brahman ank, the biggest digit. Therefore, the biggest number of each class shall have 9 at the unit place. 49 is the biggest number of forty and its class.

99 is the biggest number of ninety and its class.

Ekadhika of 99 is 100. Therefore, 100 is greater than 99 or 100 > 99.

Also 100 > 79   100 > 68

100 > 9   100 > 16

99 is less than 100 or 99 < 100.

99 < 111   68 < 100.

9. ADDITION USING PURAK

Vedic Mathematics provides multiple choice for what ever we want to do.

VM makes Maths a play.

We all like to play.

We smile when we play.

Let us learn to smile.

We have learnt how to add numbers.

Let us learn to add numbers using purak.

The best friend behaves in opposite way so while adding digits, when we use purak, it is subtracted.

Whenever we use purak we say Shudha and put a dot on the left hand side of the digit whose purak is used.

In this Vedic method of addition we use the purak of the digits. The purak we know behaves in a opposite way (because of the bar). In actual working we can either use the purak digit and do the reverse operation (of addition or subtraction) or use the purak (with bar -Rekhank) which when added is subtracted or vice versa.

Example 1:- Working steps

$$\begin{array}{rl} 7 & \\ \cdot 9 & 7 - 1 = 6 \text{ OR } 7 + \bar{1} = 6 \\ \cdot 8 & 6 - 2 = 4 \text{ OR } 6 + \bar{2} = 4 \\ + & \\ \hline 24 & \\ \hline \end{array}$$

We go on adding digits but as soon as we find that the addition of the next digit shall give a number greater than 9,we say shudha, put a dot on left hand side of digit and use its purak. The Purak digit is subtracted from the other number.

Step 1: Adding 7 to 9 shall cross 9, say shudha, put a dot on the left of 9. Purak digit of 9 is 1. So 1 is subtracted from the other number 7, we get 7 - 1 = 6.

Or purak of 9 is written as $\bar{1}$ which

when added to 7 gives us $7 + \bar{1} = 6$. We know when a bar digit (Rekhank) is added we get the difference.

Step 2: Adding 6 to next digit 8 shall cross 9, say shudha, put a dot on the left of 8. Purak digit of 8 is 2. So 2 is subtracted from 6, we get $6 - 2 = 4$. So 4 is the resulting digit.

Or purak of 8 is written as $\bar{2}$ which when added to 6 gives us $6 + \bar{2} = 4$. However it shall be more convenient to go on subtracting the purak digit. This direct operation of bar digits shall be used later on.

Step 3: At the tenth place, count the dots of shudha. We have 2, so 2 is the tenth place digit .

DISCUSSION

Whatever we want to do in life, most of the times we have lot of choices in doing the same. Vedic Maths system being the most natural way of working, also provides several choices at almost every step. One of these is of course, the easiest. Therefore systematic and regular study and practice of Vedic Maths trains us to be attentive and vigilant, which naturally leads to an alert and agile mind.

Example 2:

$$\begin{array}{ccc} & 3 & 6 \\ & 6 & 2 \\ \overset{\cdot}{+} & \dot{5} & 4 \\ \hline 1 & 5 & 2 \\ \hline \end{array}$$

Step 1: 6 + 2 = 8; For adding 8 to 4 we obviously have two choices.

Moving sequentially, we say shudha, put a dot on left of 4. Purak digit of 4 is 6 and 8 - 6 = 2.

OR The purak of 4 is $\bar{6}$ so 8 + $\bar{6}$ = 2.

However, it shall be much more convenient, easy and simple to use the purak digit of the bigger digit 8 (which is 2) and get 4 - 2 = 2. Since we have used the purak of one of the digits we say Shudha and put a dot on left of the digit being added.

NOTE: We can clearly see that while adding two digits it shall be convenient to use the purak of the bigger digit.

Step 2: 3 + 6 = 9; 9 is to be added to 5 (which has a dot above it) we can clearly see three alternative ways of doing this.

Most convenient shall be to take Ekadhika of 9 just obtained (due to the presence of dot), 9 + 1 = 10, say shudha, put a dot on the left and leave 1 at tenth place. Further 0 + 5 = 5.

OR We could take the purak digit of 9 (which is 1), say shudha, put a dot on left of 5 and 1 - 1 = 0 (the first one being the value of the dot above 5). Further, 0 + 5 = 5.

OR As a most inconvenient alternative we can take Ekadhika of 5 (since 5 has a dot above it), 5 + 1 = 6. Say shudha, put a dot on left of 5, purak digit of 6 is 4. Therefore 9 - 4 = 5 is the resulting digit (here again it

shall be slightly more convenient to use the Purak of bigger digit 9 and then 6 - 1 = 5).

Step 3: Move to hundredth place and on counting the number of dots of shudha, we get 1.

Therefore 152 is the answer.

We can see in step 2, the different alternatives provided by the Vedic Maths system. Further, it is important to note that initially even if we fail to select the most efficient sequence of steps, we shall still get the correct answer.

DISCUSSION:

The teachers are requested to carefully prepare lot of practice problems till the children are very confident of the different alternative ways. It shall be a quite blissful experience to write a problem on the blackboard and invite the students to explain different alternative ways of solving the same problem, and then to ask the whole class to select the most convenient steps to be followed. This practice shall systematically train the young minds to think of several alternatives for handling any problem which in turn shall be very effective for the better overall mental growth. It shall also give the child a sense of achievement as she can independently give another sequence for solving the problem. This naturally converts Maths into a playful subject which would frequently result in bubbling of joy and bliss in the Vedic Maths classes at all levels without exception.

Vedic Maths system puts us on the path of infinite

possibilities. The process of learning the addition of numbers does not stop here. It rather opens up. As we practice we shall start looking for patterns of digits being added which shall further improve the working efficiency. Further two, three, four or even more digits can be added simultaneously.

10. SUBTRACTION USING PURAK

Example 1:

$$\begin{array}{rcc} & 1 & 8 \\ - & \overset{\bullet}{} & 9 \\ \hline & 0 & 9 \\ \hline \end{array}$$

Step 1: (a) The lower digit 9 at the unit place is bigger, say shudha and put a dot on left of 9.

(b) The purak of 9 is 1 which is added to the above digit 8 as purak behaves in opposite way. Therefore 8 + 1 = 9 is the resulting digit.

Step 2: Move to tenth place. We have a dot at the lower place which we know is equal to 1. This is equal to the above digit 1. Therefore 1 - 1 = 0. Hence 09 is the result.

NOTE:- Subtraction using bar digits may be taught only after the child has nicely practiced the method of directly adding the purak digit to the above digit.

Example 2:

$$\begin{array}{rcc} & 4 & 5 \\ - & \overset{\bullet}{2} & 7 \\ \hline & 1 & 8 \end{array}$$

Step 1: (a) The lower digit 7 is bigger, say shudha and put a dot on left digit. Purak of 7 is 3, which is written as $\bar{3}$.

(b) Add the Purak to the above digit 5. So 5 + 3 = 8. OR 5 - $\bar{3}$ = 8 as we know that when we subtract a bar digit from a positive digit it is added.

Step 2: Move to tenth place. Ekadhika of lower digit 2 is 3 (since it has a dot above it). Further 4 - 3 = 1. Hence 18 is the answer.

In similar way we can subtract even 3 digit numbers very conveniently and directly.

11. MULTIPLICATION

Add three 5 times.

3 + 3 + 3 + 3 + 3 = 15

Adding the same number many times takes a long time. As an alternative we can also find the sum by multiplying 3 with 5.

3 x 5 = 15. We can do this quickly if we remember the multiplication table of 3. We can also write this in a column.

$$\begin{array}{r} 5 \\ \times\ 3 \\ \hline 1\ 5 \\ \hline \end{array}$$

Here we are using the third Vedic sutra Urdhva Tiryagbhyam which means vertically and crosswise. In multiplying two single digits we use the first part of sutra and multiply vertically.

12. VINCULUM NUMBERS

A number may have both positive and negative digits.

The bar digits are called Rekhank (Vinculum digits). In general, a number with both positive and negative digits is called a Vinculum number.

We know that a digit with a bar (Rekhank) is negative digit and a digit without bar is a positive digit. Let us learn Vinculum numbers.

Any number can be written in Vinculum form. Any number can be written in many forms. So in Vedic Maths we can play with numbers but each form shall always have the same value.

Example : $8 = 10 - 2 = 1\bar{2}$

Eight 8, we can also write as $1\bar{2}$. Here, 1 at the tenth place has a value equal to 10 and 2 (Rekhank 2) at the unit place is a negative digit and has value equal to -2. Therefore $1\bar{2} = 10 - 2 = 8$. The value of $1\bar{2}$ is also equal to 8. But this is the second way of writing the digit 8. Further, we remember that purak digit of 8 is 2, purak is written as $\bar{2}$ (Rekhank 2) and $8 = 1\bar{2}$. Similarly, let us write each digit in second form using its purak.

$9 = 10 - 1 = 1\bar{1}$

$8 = 10 - 2 = 1\bar{2}$

$7 = 10 - 3 = 1\bar{3}$

13. ADDITION OF VINCULUM NUMBERS

Rules: 1 A positive digit added to a positive digit shall

result in addition of the two digits.

$2 + 3 = 5$

2. A positive digit added to a Rekhank bar digit (negative digit) shall result in subtraction of the two digits.

The sign of the result shall be the same as that of the bigger digit.

$4 + \bar{2} = 2$ $\quad 5 + \bar{3} = 2$ $\quad \bar{2} + 6 = 4$

$3 + \bar{2} = 1$ $\quad 6 + \bar{4} = 2$ $\quad \bar{3} + 5 = 2$

Here bigger digit is positive so result has no bar.

$3 + \bar{4} = \bar{1}$

$1 + \bar{4} = \bar{3}$

$2 + \bar{3} = \bar{1}$

$\bar{5} + 3 = \bar{2}$

$\bar{4} + 1 = \bar{3}$

Here the bigger digit has a bar, is Rekhank, so result also has a bar - is Rekhank.

3. A Rekhank (bar digit) is added to another Rekhank (bar digit) shall result in addition of the two digits. And the result shall be Rekhank (with a bar).

$\bar{2} + \bar{3} = \bar{5}$

$\bar{1} + \bar{2} = \bar{3}$

$4 + \bar{1} = \bar{5}$

14. MULTIPLYING TWO DIGIT NUMBERS
(Without carrying)

Let us learn the third Vedic sutra. It is

Urdhvatiryagbhyam. It means Vertically and Crosswise. This sutra is used for multiplication. There are two steps of working. Since we are multiplying only by a single digit number, we shall use only the first part of the sutra.

We multiply vertically and get the first part of the answer. Shift the lower digit on the left hand side, multiply vertically (simplify using Urdhva process) to get the second part of the answer.

Multiply 23 by 2.

Step 1: We multiply the unit digits vertically

2 x 3 = 6 ones

Put 6 in one's column.

$$\begin{array}{cc} T & O \\ 2 & 3 \\ \times & 2 \\ \hline 4 & 6 \\ \hline \end{array}$$

Step 2: Shift the lower digit 2 below the tenth digit 2 and multiply vertically.

2 x 2 = 4 tens.

$$\begin{array}{r} 2 \\ \times\ 2 \\ \hline 4 \\ \hline \end{array}$$

Put 4 in the ten's column.

Alternative Method: Vedic maths also provides us alternative methods of working. The Urdhva procedure of working which we have just learnt is most convenient when we are multiplying with a single digit number.

We may also like to learn the alternative method of working in which we shall use both parts of the Urdhva sutra.

First step is based on first part of the sutra. The

first part says Urdhva, it means vertically. So we multiply the unit digits vertically. The second part of the sutra has Tiryak which means crosswise. Here we multiply the digits crosswise.

Example: Multiply 23 by 2.

Step 1: We multiply the unit digits vertically.

$$\begin{array}{cc} 2 & 3 \\ \times & 2 \\ \hline 4 & 6 \\ \hline \end{array}$$

OR 2 x 3 = 6 ones

Put 6 in one's column.

Step 2: Second part of sutra contains tiryak, which means crosswise. So we multiply crosswise.

Multiply 2 tens by 2.

2 x 2 = 4 tens

Put 4 in ten's column.

DISCUSSION:

The lower single digit number is essentially a two digit number with 0 on the left hand side. The complete working of the second step shall actually include two cross products and their addition.

$$\begin{array}{ccc} & 2 & 3 \\ \times & 0 & 2 \\ \hline & 4 & 6 \\ \hline \end{array}$$

Step 2: Crosswise means

2 x 2 + 0 x 3 = 4 + 0 = 4

Since we know that the contribution of the second part of the cross product shall always be zero (as lower single digit number shall always have zero as the left digit) we need not tell this part to the child in the beginning. But it is important for us to know this. When the child has nicely practised the

Step 3: Shift the lower digit 3 below the next digit (hundredth digit) and multiply vertically.

$$\begin{array}{r} 2 \\ \times\ 3 \\ \hline 6 \\ \hline \end{array} \qquad \text{OR} \qquad 3 \times 2 = 6$$

Put 6 in hundredth column.

16. ADDITION AND SUBTRACTION OF MONEY

The Vedic upsutra shudha shall be used in each case. The steps to be followed are exactly the same as already explained in addition and subtraction of ordinary numbers.

Example 1: Add following paisa:

$$\begin{array}{rcc l} & 1 & 3 & \text{paisa} \\ & \dot{1} & 9 & \text{paisa} \\ + & 5 & 4 & \text{paisa} \\ \hline & 8 & 6 & \text{paisa} \\ \hline \end{array}$$

Step 1: Unit place. For adding 3 to 9, we say shudha, put a dot on left of 9 (above 1). The purak of 9 is $\bar{1}$, so $3 + \bar{1} = 2$.

OR subtract the purak digit of 9 (which is 1) from 3, 3 - 1 = 2; Further , 2 + 4 = 6.

Step 2: 1 + 1 + 1 (due to dot) = 3,

3 + 5 = 8.

Hence 86 paisa is the answer.

multiplication by one digit we can also explain the complete details. Further, details shall come in next Ankur.

15. MULTIPLYING THREE DIGIT NUMBERS :
(Without carrying)

Example 1: Multiply 231 by 3.

We use the Urdhva sutra for multiplication. Since we are multiplying only by a single digit number. We can do this by using only the first part of the sutra.

The first part of the sutra says Urdhva, which means Vertically.

The steps of working are simple and straight forward. We multiply vertically the unit digits and get the first part of the result. Shift the lower digit on the left hand side and multiply vertically the digits to get the second part of the result. Once again shift the lower digit below the third digit and multiply vertically, to get the third part of the result.

$$\begin{array}{r} 1 \\ \times\ 3 \\ \hline 3 \\ \hline \end{array}$$

Step 1: Multiply the unit digits vertically. 3 x 1 = 3. Put 3 in one's column.

$$\begin{array}{ccc} H & T & O \\ 2 & 3 & 1 \\ & \times & 3 \\ \hline 6 & 9 & 3 \\ \hline \end{array}$$

Step 2: Shift the lower digit 3 on the left hand side (below the next digit 3) and multiply vertically.

$$\begin{array}{rr} & 3 \\ \times & 3 \\ \hline & 9 \\ \hline \end{array}$$

OR 3 x 3 = 9

Put 9 in the ten's column.

Example 2:

$$\begin{array}{ccccccccc} & 3 & 7 & 5 & \text{rupees} & 2 & 9 & \text{paisa} \\ + & \dot{2} & \dot{8} & 7 & \text{rupees} & \dot{3} & 7 & \text{paisa} \\ \hline & 6 & 6 & 2 & \text{rupees} & 6 & 6 & \text{paisa} \\ \hline \end{array}$$

LECTURE -2

1. INTRODUCTION

In previous Ankurs we have studied the simple Vedic Methods of solving Maths problems. We have also studied the unique Vinculum number system. Vinculum approach makes it possible to write any single digit number in two alternative forms. Like

$$8 = 10 - 2 = 1\bar{2}$$

$$4 = 10 - 6 = 1\bar{6}$$

Therefore, we have complete freedom to use numbers in any convenient form. So, we can freely play with numbers. We have already learnt simple operations of Vinculum numbers.

2. SUBTRACTION OF VINCULUM NUMBERS

Subtraction of vinculum numbers is an interesting topic to be learnt carefully and slowly. Vedic maths provides many choices. The subtraction of vinculum numbers can most easily be learnt by coverting it to the problem of addition. The only rule to be remembered is that the sign of the digit being subtracted is changed and then added to the other digit.

Now let us learn the step by step process of subtraction of Vinculum numbers.

Step 1: Change the sign of the number to be subtracted.

Step 2: Add the two numbers digit by digit.

Case I: A bar digit to be subtracted from a positive digit.

$2 - \bar{1} = 2 + 1 = 3$

Step 1: We change the sign of the number to be subtracted, here $\bar{1}$.

Since it has a bar, we remove the same and get 1.

Step 2: Add the two numbers.

So $2 + 1 = 3$.

$4 - \bar{2} = 4 + 2 = 6$; $5 - \bar{3} = 5 + 3 = 8$

$3 - \bar{1} = 3 + 1 = 4$; $7 - \bar{2} = 7 + 2 = 9$

$3 - \bar{3} = 3 + 3 = 6$; $7 - \bar{4} = 7 + 4 = 11$

Case II: A positive digit (without bar) to be subtracted from a bar digit.

$\bar{2} - 1 = \bar{2} + \bar{1} = \bar{3}$

Step 1: Change the sign of the number to be subtracted, here 1. Since the digit is without bar so we put a bar and get $\bar{1}$.

Step 2: Add the two numbers.

Here $\bar{2} + \bar{1} = \bar{3}$ (by rule 2 of addition). Since both the digits are with bar, the result shall be the addition of the two digits and shall be with a bar.

Similarly

$\bar{1} - 2 = \bar{1} + \bar{2} = \bar{3}$; $\bar{1} - 3 = \bar{1} + \bar{3} = \bar{4}$

$\bar{3} - 3 = \bar{3} + \bar{3} = \bar{6}$; $\bar{4} - 1 = \bar{4} + \bar{1} = \bar{5}$

$\bar{5} - 4 = \bar{5} + \bar{4} = \bar{9}$; $\bar{7} - 4 = \bar{7} + \bar{4} = \overline{11}$

Case III: A positive digit (without bar) to be subtracted from a positive digit.

$5 - 3 = 5 + \bar{3} = 2$

Step 1: Change the sign of the number to be subtracted, here 3. Since the digit is without a bar, so we put a bar and get $\bar{3}$.

Step 2: Add the two numbers.

So $5 + \bar{3} = 2$ (by rule 3(a)) since only one digit has a bar, we take the difference.

Similarly,

$5 - 4 = 5 + \bar{4} = 1$; $7 - 5 = 7 + \bar{5} = 2$

$6 - 1 = 6 + \bar{1} = 5$; $8 - 7 = 8 + \bar{7} = 1$

$9 - 6 = 9 + \bar{6} = 3$; $9 - 9 = 9 + \bar{9} = 0$

Case IV: A bigger positive digit subtracted from a smaller positive digit (without bar).

$3 - 4 = 3 + \bar{4} = \bar{1}$

Step 1: Change the sign of the digit to be subtracted, here 4. Since the digit is without bar, we put a bar and get $\bar{4}$.

Step 2: Add the two digits.

$3 + \bar{4} = \bar{1}$ (by rule 3(b)) since only the bigger digit has a bar, we take the difference and the result shall be with bar.

Similarly,

$4 - 5 = 4 + \bar{5} = \bar{1}$; $5 - 7 = 5 + \bar{7} = \bar{2}$

$1 - 6 = 1 + \bar{6} = \bar{5}$; $7 - 8 = 7 + \bar{8} = \bar{1}$

$7 - 9 = 7 + \bar{9} = \bar{2}$; $7 - 7 = 7 + \bar{7} = 0$

Case V: A bar digit (negative) to be subtracted from a smaller bar digit.

$\bar{1} - \bar{3} = \bar{1} + 3 = 2$

Step 1: Change the sign of the number to be subtracted, here $\bar{3}$. Since the digit has a bar, so we remove it and get 3.

Step 2: Add the two digits.

Here $\bar{1} + 3 = 2$ (by rule 3(a)). Since only one of the digits has a bar so the result shall be the difference of the two. Further, since the bigger digit is without bar, the result shall also be without bar.

Similarly,

$\bar{2} - \bar{5} = \bar{2} + 5 = 3$; $\bar{3} - \bar{4} = \bar{3} + 4 = 1$

$\bar{5} - \bar{7} = \bar{5} + 7 = 2$; $\bar{1} - \bar{2} = \bar{1} + 2 = 1$

$\bar{3} - \bar{8} = \bar{3} + 8 = 5$; $\bar{6} - \bar{6} = \bar{6} + 6 = 0$

Case VI: A smaller bar digit to be subtracted from a bigger bar digit.

$\bar{2} - \bar{1} = \bar{2} + 1 = \bar{1}$

Step 1: Change the sign of the digit to be subtracted here $\bar{1}$. Since the digit has a bar, so we remove it and get 1.

Step 2: Add the two digits.

So $\bar{2} + 1 = \bar{1}$ (by rule 3(b)). Since only one of the digits has a bar, so the result shall be the difference of the two. Further, as the bigger digit is with bar (negative), the result shall also have a bar.

Similarly,

$\bar{3} - \bar{1} = \bar{3} + 1 = \bar{2}$; $\bar{2} - \bar{2} = \bar{2} + 2 = 0$

$\bar{5} - \bar{3} = \bar{5} + 3 = \bar{2}$; $\bar{7} - \bar{4} = \bar{7} + 4 = \bar{3}$

$\bar{6} - \bar{5} = \bar{6} + 5 = \bar{1}$; $\bar{9} - \bar{6} = \bar{9} + 6 = \bar{3}$

DISCUSSION: (i) Many of us are very likely to feel that the operation of bar digits is very difficult and hence should not be introduced so early. But some idea of this should be given with very carefully selected problems.

(ii) Some weak children may find this little confusing and we should not worry if they cannot solve all the problems. But it is important to give them the training of both positive and bar digits.

(iii) This difficulty may be specially felt if we are directly introducing Vedic Maths in class -III. The answer is to go slow.

(iv) It is most important that only one rule should be taught on one day and lot of practice given till the children are confident about the same. Simple practice is the sure solution of all the difficulties faced.

(v) It shall be useful to quote the feed back from America. "Vedic Math is good. It takes a long time for me to learn it, but once I figure it out it is easier" (A student very slow to learn math in the past. He says he has never enjoyed math).

"It is a lot easier. It seems complicated at first, but when you begin to catchon, it is a lot faster. It ismuch easier to compute bigger numbers. Vedic Math is a lot better".

3. BIGGER VINCULUM NUMBERS

We can convert bigger numbers into vinculum form. For converting any number to vinculum form we make use of first two sutras.

$98 = 100 - 02 = 1\ 0\ \bar{2}$

$1\ 0\ \bar{2}$ is the vinculum form of number 98. Here, we know that 1 at the hundredth place has a numerical value of 100 and $\bar{2}$ at the unit place has a value of -2.

Therefore, $1\ 0\ \bar{2} = 100 - 2 = 98$.

But, operating $10\bar{2}$ is much more simpler easier and convenient than using 98. For converting bigger digits of any number to vinculum form we proceed as follows:

Step 1: Identify the group/groups of digits to be converted to vinculum form and compute their purak (complement) using the second Vedic sutra. Nikhilam Navatascaramam Dasatah( ).All from 9 last from 10. Purak of each digit is computed from 9 and purak of the last digit from 10. These purak digits are written with a bar as usual.

Step 2: Further we use the first Vedic sutra. Ekadhikena Purvena ( )means One more than the previous One. We take Ekadhika of the digit just before the group of digits whose purak has been just obtained.

Let us take a few examples.

Example 1:

1 8 8

Step 1: We may like to convert the last two digits (88) into vinculum form. Invoking Nikhilam sutra, we get the purak digits as 9 - 8 = 1 and last from ten ,

10 - 8 = 2

Step 2: Ekadhika of 1 (the previous digit) is 1 + 1 = 2.

Therefore

$1\ 8\ 8 = 2\ \bar{1}\ 2$

We can also understand this as

$2\ \bar{1}\ \bar{2} = 200 - 10 - 2 = 188$

Example 2:

$8\ 9 = 0\ 8\ 9$

Step 1: Puraks are $9 - 8 = 1$ and $10 - 9 = 1$

Step 2: [illegible] 1

Therefore $8\ 9 = 1\ \bar{1}\ \bar{1}$

Note: The teachers are very strongly requested to give lot of practice in using bar digits. So that the children can handle them as freely as positive digits. Once the simple operational rules of bar digits are understood and well practised the entire computational effort in Vedic maths system is drastically cutdown to their bare minimum essential level. This makes maths a playful subject and each sum is then a simple smiling task.

4. CHECK METHOD OF VINCULUM NUMBERS

The Vedic sutra number 15 (Gunitasamuccayah) gives us a complete checking system for most of arithmetic after adding subtracting, multiplying or converting to vinculum form we may like to check the correctness of our working.

Let us learn the simple steps of checking the Vinculum numbers. The procedure is simple.

Step 1: We convert the number to a single digit.

Step 2: We also convert the vinculum number into a single

digit. If the second digit is a negative digit, we normalize it by adding 9.

Step 3: Our conversion to vinculum form is correct then the digit obtained in Step (1) and (2) should be the same.

Example 1:

$$4\ 6\ 9 = 5\ \bar{3}\ \bar{1}$$

Step 1: 4 + 6 = 10, as soon as we get two digits, we again add the digits of result to get a single digit. Here, we add 1 and 0 of 10, so 1 + 0 = 1. Further, 1 + 9 = 10, and once again 1 + 0 = 1. Finally, by adding all the digits we have got 1.

Alternatively, we could also add from right to left. 9 + 6 = 15, so 1 + 5 = 6. Further, 6 +4 = 10, so 1 + 0 = 1. We shall always get the same single digit whether we add from left to right or right to left.

Step 2: Similarly, we add the digits of vinculum number.

From left $5 + \bar{3} = 2$, and $2 + \bar{1} = 1$

or from right $\bar{1} + \bar{3} = \bar{4}$ and $\bar{4} + 5 = 1$.

Step 3: We have obtained the same digit 1 in step 1 and step 2. So our conversion is Okay.

5. ADDITION AND SUBTRACTION OF 4 OR 5 DIGIT NUMBERS

The addition or subtraction of bigger digits can also be carried out very easily using the same Vedic upsutra Shudha. In actual working column by column the different alternatives shall be always available, any one of which can be used. (Refer Ankur 4).

6. CHECKING OF ADDITION AND SUBTRACTION

The Vedic Sutra Gunit Sammuccayah can also be used for checking the sums of addition and subtraction. The steps to be followed are similar to those of the Vedic check method of Vinculum numbers.

Example 1: Check the sum

$$\begin{array}{r} 1\ 8\ 3 \\ +\ \dot{3}\ 3\ 4 \\ \hline 5\ 1\ 7 \\ \hline \end{array} \qquad \begin{array}{r} 3 \\ +\ 1 \\ \hline 4 = 4 \\ \hline \end{array}$$

Step 1: 1 + 8 = 9, 9 + 3 = 12 so 1 + 2 = 3.

Step 2: 3 + 3 = 6, 6 + 4 = 10, so 1 + 0 = 1.

Step 3: 3 + 1 = 4 (Add digits of step 1 & 2)

Step 4: 5 + 1 = 6, 6 + 7 = 13, so 1 + 3 = 4

Step 5: 4 = 4 (step 3 & 4)

Hence result is Okay.

7. CONVERSION TO ORDINARY FORM

We have learnt how we can convert the bigger digits of any number into vinculum form. For this task we use the first Ekadhiken sutra along with the second Nikhilam sutra. How can we convert a Vinculum number into all positive digit number. For this conversion once again we use the Nikhilam sutra to get the purak of bar digits. Further, the digit just before the bar digits is reduced by 1 by using the Ekanyunena Purvena(   ) sutra number 14 which means 'One less than the previous one'.

Example 1: Convert $3\bar{2}\bar{1}$.

Step 1: The group of last two digits has a bar, so we compute their purak using Nikhilam sutra, all from nine last from ten.

$9 - 2 = 7$, $10 - 1 = 9$.

Step 2: Further, using the Ekanyunena sutra, we lessen the previous number 3 by 1. So $3 - 1 = 2$. Therefore

$$3\,\bar{2}\,\bar{1} = 2\,7\,9$$

The result can again be checked by using the simple Vedic check method.

CHECK

Step 1: $3 + \bar{2} = 1$, $1 + \bar{1} = 0$

Since we have got 0, so we add 9 and get $0 + 9 = 9$.

Step 2: $2 + 7 = 9$, $9 + 9 = 18$ so $1 + 8 = 9$.

The digits obtained in step 1 and 2 are equal so our result is Okay.

8. CONVERSION INTO VINCULUM WITHOUT NIKHILAM

We have learnt how to convert the given number into vinculum form using Ekadhiken sutra alongwith Nikhilam sutra. We can also get the bigger digits converted to vinculum form directly by using the sutra( ) Sankalana vyavakalanabhyam which means by addition and subtraction.

Example 1: Convert 9 to Vinculum form

The Sankalana vyavnakalanabhyam sutra consists of two parts:

Part 1: By addition : So we add the number to itself.

$9 + 9 = 18$

Part 2: The second part of the sutra is by subtraction, so we subtract number from the previous result without using shudha.

$$\begin{array}{r} 1\ 8 \\ -0\ 9 \\ \hline 1\ \bar{1} \end{array} \qquad \text{as} \qquad \begin{array}{l} 8 - 9 = \bar{1} \\ 1 - 0 = 1 \end{array}$$

Therefore $9 = 1\,\bar{1}$

Example 2:

$$\begin{array}{r} 1\ 9 \\ +\ \dot{1}\ 9 \\ \hline 3\ 8 \\ \hline \end{array} \qquad \begin{array}{r} 1\ 7 \\ +\ \dot{1}\ 7 \\ \hline 3\ 4 \\ \hline \end{array}$$

Further,

$$\begin{array}{r} 3\ 8 \\ -\ 1\ 9 \\ \hline 2\ \bar{1} \\ \hline \end{array} \qquad \begin{array}{r} 3\ 4 \\ -\ 1\ 7 \\ \hline 2\ \bar{3} \\ \hline \end{array}$$

9. URDHVA SUTRA

Let us learn the third Vedic Sutra. It is (Urdhva-tiryagbhyam). It means vertically and cross-wise. This sutra can be used for multiplying any number with another number. In the previous Ankur we have already studied the use of Urdhva sutra for multipllication by a single digit. Now we shall study the complete details of the Urdhva method of multiplicaion. Let us learn the method of simplification.

Single column:

If we have to simplify two digits in a single column we multiply vertically.

Example : 2   R = 2x3 = 6

x 3

--

6

--

Two columns:

If we have to simplify four digits in two columns we multiply cross-wise.

Example: 2 1   R = (1 x 1) + (3 x 2)

x 1 3   = 1 + 6

----   = 7

Three columns:

If we have to simplify six digits in 3 columns, we multiply the corner digits crosswise and the middle column digits vertically. The sum of the three products gives us the desired value.

Example 1: 1 2 1   R = 1 x 3 + 1 x 4 + 2 x 1

4 1 3

-----   = 3 + 4 + 2 = 9

PROCEDURE: For multiplying two numbers we use the following steps:

Step 1: Simplify the first column

Step 2: Simplify two columns

Step 3: Leave the first column and simply the remaining column

Each step gives us three parts of the answer.

Example 1: Multiply 12 x 13

The answer shall have three parts. So we draw two slashes to mark three spaces.

```
  1 2
x 1 3
------
 / /
------
```

Step 1: Simplify the first column. We multiply vertically. 2 x 3 = 6, and write this on the right side space.

```
(1)  1 2        (2)  1 2        (3)  1 2
   x 1 3           x 1 3           x 1 3
   ----            ----            ------
   / / 6           /5/6            1/5/6
   ------          -----           ------

                                = 156
```

Step 2: Simplify two columns. We multiply cross wise and add. SO (1 x 3) + (1 x 2) = 3 + 2 = 5

Write this in the middle space .

Step 3: Leave the first column .We are left with only one column. Simplify the left single column vertically.

1 x 1 = 1.

We write this in the left side space.

Hence 156 is the answer.

Example 2: Multiply 21 by 14

```
     2 1        2 1          2 1
   x 1 4    x   1 4       x  1 4
   -----      ------      -------

 /  /  4      /9/4        2/9/4     =  294
   -----      ------      -------
```

Step 1: Simplify first column vertically 4 x 1 = 4.

Step 2: Simplify two columns crosswise.

(2 x 4) + (1 x 1) = 8 + 1 = 9

Step 3: Leave the first column simplify the left over single column vetically $2 \times 1 = 2$.

Answer is 294.

Example 3: Multiply 22 x 13

$$\begin{array}{c} 2\ 2 \\ \times\ 1\ 3 \\ \hline /\ /\ 6 \\ \hline \end{array} \qquad \begin{array}{c} 2\ 2 \\ \times\ 1\ 3 \\ \hline /\ 8\ /6 \\ \hline \end{array} \qquad \begin{array}{c} 2\ 2 \\ \times\ 1\ 3 \\ \hline 2/8/6 \\ \hline \end{array} = 286$$

10. MULTIPLICATION OF BAR DIGITS

We have learnt the use of bar digits and also their addition and subtraction. We have seen the simplification provided by the use of bar digits in addition and subtraction. The use of vinculum numbers drastically reduces the computational effort in multiplication, division etc. Now let us learn the simple rules of multiplication of bar digits.

Rule 1: If a positive digit is multiplied with another positive digit, the result is positive (without bar).

$$2 \times 3 = 6$$

Rule 2: If a bar digit is multiplied with another bar digit, the product shall have no bar (the result shall be a positive number).

$$\bar{2} \times \bar{4} = 8 \ ; \quad \bar{3} \times \bar{4} = 12$$

Rule 3: If only one of the digits has a bar, all the digits of the product shall be with bar (the result shall be negative)

$$2 \times \bar{3} = \bar{6} \ ; \quad \bar{4} \times 3 = \bar{1}\,\bar{2}$$

Example 1:

(i) Multiply $1 \times 2 = 2$ as no digit has a bar.. So result is positive and has no bar.

(ii $\bar{2} \times 2 = \bar{4}$ as only one (first) digit has a bar, so product shall be with bar.

(iii) $3 \times \bar{2} = \bar{6}$ as only one (here second) digit has a bar, the result shall be with bar.

(iv) $\bar{4} \times \bar{5} = 20$ as both digits have a bar, the result shall be positive.

(v) $0 \times \bar{3} = 0$ as 0 multiplied with any number gives 0.

Exercise : Multiply

$3 \times 4 =$ ------ ; $\bar{2} \times 1 =$ -----

$2 \times \bar{4} =$ ------ ; $\bar{2} \times \bar{3} =$ -----

$1 \times \bar{6} =$ ------ ; $\bar{7} \times 2 =$ ------

$3 \times \bar{5} =$ ------ ; $\bar{5} \times \bar{6} =$ -----

$\bar{4} \times 8 =$ ------ ; $7 \times \bar{3} =$ -----

$2 \times \bar{1} \times \bar{2} =$ ------

11. NIKHILAM MULTIPLICATION

We have learnt the second Vedic sutra (Nikhilam Navatascaram Dastah) means all from nine last from Ten. Nikhilam method of multiplication can be used for multiplying numbers with big digits.

Example 1: Multiply 9 x 6.

Step 1: Select nearest base. Here base is 10. Base is the number with left digit as 1 and all other digits as 0.Therefore 10, 100 , 1000, 10000 etc. are different bases.

$$\begin{array}{c} 9 - 1 \\ 6 - 4 \\ \hline 5\ /\ 4 \\ \hline \end{array}$$

Step 2: Compute the deviation of each number from the base. Write the deviation with sign in front of the number as shown. We use Nikhilam sutra. Deviation of 9 is 1 (last from ten 10 - 9). Since 9 is smaller than the base 10, we write this with the negative sign. Similarly,deviation of 6 is 4. (10 - 6 = 4) .Which is the same as the purak of the last digit from 10. Since 6 is smaller than the base, we write the deviation with a negative sign.

Step 3: Divide the result space by drawing a slash.

The right hand side of the result is simply the multiplication of the two deviations. Therefore $\bar{1} \times \bar{4} = 4$. This is written on the right hand side of the slash.

Step 4: The left hand side of the result is simply the addition of any number with the deviation of the second number.

(i) Therefore $6 + \bar{1} = 5$

Or

(ii) $9 + \bar{4} = 5$

$$\begin{array}{c} 9 - 1 \\ 6 - 4 \\ \hline 5\ /\ 4 \\ \hline \end{array} = 54$$

The resulting 5 is written on the left hand side.

Hence 54 is the answer.

Example : Multipy 8 x 9 .

Step 1: Base is 10.

Step 2: Compute the deviations.

The puraks are $\bar{2}$ & $\bar{1}$.

$$\begin{array}{c} 8 - 2 \\ 9 - 1 \\ \hline 7\ /\ 2 \end{array} = 72$$

Step 3: Multiply the deviations $\bar{2}$ and $\bar{1}$ so $\bar{2} \times \bar{1} = 2$ (R.H.S.)

Step 4: Cross add $8 + \bar{1} = 7$ or $9 + \bar{2} = 7$ (L.H.S.).

Hence 72 is the result.

12. VEDIC CHECK METHOD FOR MULTIPLICATION

The Vedic sutra Gunit Sammuccaya can also be used for checking the product of multiplication of the numbers.

Steps to be followed are to be similar to those of the Vedic Check method of addition and subtraction.

Example 1: Check the product. 18 * 11 = 198.

In Vedic system we multiply these two numbers eiter by using Urdhva method or Nikhilam method.

Step 1: Convert the above number into a single digit. Here 1 + 8 = 9. Write this in the upper space.

Step 2: Now convert the multiplier lower number into a single digit. Here 1 + 1 = 2. Write this in the lower space.

```
     1 8
     1 1
   ---------
   1 / 9 / 8
   ---------
    = 198

  ----------
      9
  9       9
      9
  ----------
```

Step 3: (a) Multiply the single digits obtained in step 1 and 2. So 9 x 2 = 18.

(b) Convert the obtained product into a single digit. So 1 + 8 = 9. Write this in the right hand space.

Step 4: Convert the result into a single digit. Hence 1 + 9 = 10, so 1 + 0 = 1, 1 + 8 = 9. Write this on the left hand space.

Step 5: The digits obtained in step 3 (b) and 4 are equal. (digit on right hand space = digit on left hand space). Here 9 = 9. Therefore the result has no random error.

13. MULTIPLICATION TABLES

We should remember number tables as their use greatly helps in computation work. Vedic Mathematics provides us a very simple and easy method of writing of the multiplication tables of any desired number. Vedic method is so quick that after a little practice if we have a race with a calculator Vedic maths shall always win the race. So let us learn the simple method of writing multiplication table of any number.

Example 1: Write the multiplication table of number 19.

We convert the given number into Vinculum form. This we can do by using any of the two methods already learnt. Therefore using Sankalana vyavakalanabhyam sutra by addition

19 + 19 = 38 and by subtraction (no shudha)

38 - 19 = $2\bar{1}$

Therefore $\bar{1}$ is the operator at the unit place which we shall use for computing the subsequent digits of the table at diferent levels. Similarly 2 is the operator at the tenth place and shall be used for computing the digits of different levels at the tenth place.

The process of computing the digit at next level is very simple we add the operating digit to the digit at any level and get the digit at the next level. It shall be convenient to write the operating digits above the given number.

Let us take the table of 19 as our first example. The

operator is -1 $(\bar{1})$for the unit place and +2 for the tenth place.

(i) That means we write the number 19 and divide it into two parts (may be by drawing a vertical line in between the two digits).

(ii) We apply the operator to the digit at the unit place (with appropriate sign - of course). In the present example we apply the operator -1 to the digit 9 (at the unit place) and therefore we get 9 - 1 = 8 as the second level unit number, 8 - 1 = 7 as the third level digit, 7 - 1 = 6 the fourth level, 6 - 1 = 5 as the fith level and so on, finally 1 - 1 = 0 at the tenth level. However, if required, the carry over can be easily done.

This step gives us the digits of the unity column 1.

(iii)Next we apply the operator of the tenth place to the digit at the tenth place.Presently, we add +2 at every stage to get the tenth place number at the next level. Therefore, we get 1 + 2 = 3 as the digit at the second level. Similarly moving further, 3 + 2 = 5 is the third level digit, 5 + 2 = 7 is the fourth level digit 7 + 2 = 9 the fifth level, 9 + 2 = 11 as the sixth level digit, 11 + 2 = 13 as the seventh level

2	- $\bar{1}$
1	9
	8
	7
	6
	5
	4
	3
	2
	1
	0

digit and so on, 17 + 2 = 19 at the tenth level.

Notes

1. However it is not essential to stop at tenth level as the same operators can be further applied for continuing to write the table upto hundredth level or even more if so desired.

2. As soon as the operation of addition gives the value larger than 9, the unity at the tenth place of addition (which shall be actually at the hundredth place of the final number) we can leave in the addition process and add only to the left over single digit.

level	2	$\bar{1}$
1	1	9
2	3	8
3	5	7
4	7	6
5	9	5
6	11	4
7	13	3
8	15	2
9	17	1
10	19	0

14. VEDIC DHWAJANK METHOD OF DIVISION

Discussion: As we divide the digits at each place, we may be left with a remainder. In Vedic mathematics the subtraction will be done separately as rough work. Slowly with practice more and more work can be done mentally without writing. We shall give practise in the same sequence and should encourage the child to do as much of rough work as possible only mentally. This practice shall help in the mental development of the child. However, initially she may like to write every step. We repeat the multiplication table

of the digit by which we are dividing till we get a number which crosses the dividend number. Then go a step backward and select the previous level digit as the quotient digit. The remainder is obtained by subtraction and is prefixed slightly below the next digit of the dividend.

Example 1: Divide 11 by 2.

We arrange the divisor and dividend as usual and repeat the multiplication table of 2. We know that 2 x 5 = 10 < 11 and 2 x 6 = 12 > 11.

```
2 / 11
-------
5 : 1 :
-------
Q   R
```

Therefore 5 is the quotient digit. To get the remainder we shall subtract 10 from 11. Hence 11 - 10 = 1 is the mainder. So 5 is the quotient digit and 1 is the remainder. In Vedic maths the remainder is always written after the quotient with colon on both sides.

Example 2: Divide 243 by 2.

Step 1: Divide the hundred place digit 2 by 2. Therefore 2 x 1 = 2 so 2 - 2 = 1 and we have remainder as 2 - 2 = 0. So we write 0 before 4, the tenth place digit.

```
2 / 2 4 3
     0 0
-----------
1 2 1 : 1 :
-----------
```

Step 2: The next dividend at tenth level is 04. Further 2 x 2 = 4 and we divide 4 by 2 and get 4 - 2 = 2 as the quotient digit, and remainder as 4 - 4 = 0 . Write 0 before 3, the unit place digit.

Step 3: The unit place dividend is 03. We can divide 1 times as 2 x 1 = 2 < 3.

and 2 x 2 = 4 > 3. Therefore 1 is the quotient digit and remainder 3 - 2 = 1.

Hence Q = 121 and R = 1.

15. FRACTIONS

For adding two fractions, if the denominator is same, we simply add the numerator and write it with denominator. This will give the required addition of fractions.

The Vedic method for adding fractions shall depend upon the pattrern of the problem. However, in every case the work involved is much lesser than the present unique approach.

Fractions with common denominqator: We simply add the numerators.

Example 1: Add (a) $\frac{3}{5} + \frac{2}{5}$

$$\frac{3}{5} + \frac{2}{5} = \frac{3+2}{5} = \frac{5}{5} = 1$$

(b) $$\frac{4}{7} + \frac{3}{7} = \frac{4+3}{7} = \frac{7}{7} = 1$$

(c) $$\frac{4}{9} + \frac{3}{9} = \frac{4+3}{9} = \frac{7}{9}$$

We use the upsutra Vilokanam which means just by glance. As soon as we find that the denominator of the given fraction is a common number we simply add the numerator digits and simplify.

Further details are presented in VM Ankur 4.

SUB SUTRAS OR COROLLARIES

1	आनुरूप्येण	Anurupyena
2	शिष्यते शेषसंज्ञः	Sisyate Sesasamjnah
3	आद्यमाद्येनान्त्यमन्त्येन	Adyamadyenantyamantyena
4	केवलैः सप्तकं गुण्यात्	Kovalaih Saptakam Gunyat
5	वेष्टनम्	Veslanam
6	यावदूनं तावदूनम्	Yavadunam Tavadunam
7	यावदूनं तावदूनीकृत्य वर्गं च योजयेत्	Yavadunam Tavadudunikrtya Varganca Yojayet
8	अन्त्ययोर्दशकेऽपि	Antyayordasakepi
9	अन्त्ययोरेव	Antyayoreva
10	समुच्चयगुणितः	Samuccayagunitah
11	लोपनस्थापनाभ्याम्	Lopanasthapanabhyam
12	विलोकनम्	Viloknam
13	गुणितसमुच्चयः - समुच्चयगुणितः	Gunitasamuccayah Samuccayagunitah
14	द्वन्द्वयोग	Dwandwa Yoga
15	शुद्धः	Shudha
16	ध्वजकं	Dhwajakam

-: VEDIC MATHEMATICS SUTRAS :-

1. By one more than the previous one
2. All from nine & last from ten
3. Vertically & Crosswise
4. Transpose & Apply
5. The summation is equal to zero
6. If one is in ratio, the other one is zero
7. By addition and subtraction
8. By the completion and non-completion
9. Sequential motion
10. The deficiency
11. Whole as one & one as whole
12. Remainder by the last digit
13. Ultimate and twice the penultimate
14. By one less than the previous one
15. The whole product is the same
16. Collectivity of multipliers

THE SIXTEEN SUTRAS OF VEDIC MATHEMATICS

सूत्र :-

1 एकाधिकेन पूर्वेण	1. Ekadhikena Purvena (also a corollary)
2 निखिलं नवतश्चरमं दशतः	2. Nikhilam Navatascaramam Dasatah
3 ऊर्ध्वतिर्यग्भ्याम्	3. Urdhva-tiryagbhyam
4 परावर्त्य योजयेत्	4. Paravartya Yojayet
5 शून्यं साम्यसमुच्चये	5. Sunyam Samyasamuccaye
6 (आनुरूप्ये) शून्यमन्यत्	6. (Anurupye) Sunyamanyat
7 संकलनव्यवकलनाभ्याम्	7. Sankalana-vyavakalanabhyam (also a corollary)
8 पूरणापूरणाभ्याम्	8. Puranapuranabhyam
9 चलनकलनाभ्याम्	9. Calana-kalanabhyam
10 यावदूनम्	10. Yavadunam
11 व्यष्टिसमष्टिः	11. Vyastisamastih
12 शेषाण्यङ्केन चरमेण	12. Sesanyankena Caramena
13 सोपान्त्यद्वयमन्त्यम्	13. Sopantyadvayamantyam
14 एकन्यूनेन पूर्वेण	14. Ekanyunena Purvena
15 गुणितसमुच्चयः	15. Gunitasamuccayah
16 गुणकसमुच्चयः	16. Gunakasamuccayah

APPLICATIONS OF MATHEMATICS AT THE UPPER PRIMARY LEVEL

By

J.N.Kapur

Mathematical Sciences Trust Society, C-766 New Friends Colony
New Delhi 110 65

WHY APPLICATIONS?

A student learns a subject best when he or she is convinced that the subject is interesting, relevant and useful. This statement applies equally well to every topic in his or her course. The teacher has to create the need for a topic to be learnt and then teach the topic.

In mathematics, we can have two types of motivations viz:

(i) Motivation from within Mathematics

(ii) Motivation from outside mathematics

The first motivation arises when a student uses the discovery approach and when he proceeds to recognise patterns in numbers and space. The student works on an open-ended problem, so that the solution of this problem automatically leads him to another open-ended problem and the process continues. The student becomes an amateur researcher. A large number of problems of this type have been given in three volumes of my book, 'Suggested Experiments in School Mathematics'.

Alternatively the motivation of learning mathematics can come from applications of mathematics in other school subjects, in every-day life and in science and technology.

In any case we have to teach, not classical mathematics or new mathematics but we have to teach RELEVANT MATHEMATICS, USEFUL MATHEMATICS, and above all MOTIVATED MATHEMATICS.

We have not to tell the school student that the mathematics he learns now will be useful later in his study of science and technology: we have to tell him that that the mathematics which he studies in school is useful here and now.

Mathematics has been described as a man-made universe i.e. it has been created to serve the needs of mankind. In fact, every topic in school mathematics was created for solving practical problems . In higher mathematics today, a great deal of research is being done for the sake of abstraction and generalisation and most of it has no immediate relevance for applications , though some of it is likely to be found useful later. However school mathematics is hundred percent useful mathematics. History of mathematics can be a good guide for pedagogy in mathematics.

The need for solving practical problems should motivate the learning of mathematics, and the mathematics learnt should be used to demonstrate further applications of this mathematics and these applications should themselves create the need for learning more mathematics and this process should continue ad infinitum, so that mathematics appears as a man-made need-based human subject of the greatest importance to the child and the society.

{Practical Problems or Internal Need of Mathematics}
Motivate. {Learning of Mathematics}
Motivate. {Possibility of More Applications}
Motivate {Learning of More Mathematics} . . .

A teacher should see applications of mathematics all round him and he should enable his students to see applications of mathematics all round themselves.

EXAMPLES OF APPLICATIONS FROM HISTORY OF MATHEMATICS

In India, mathematics was created for making 'havan kunds' and 'sacrifical altars' and for making calculations about motions of planets for astrological and religious purposes. In Babylonia, it was created for commercial transactions. In Egypt, it was created for redistribution of land after floods in the Nile river. In Greece, it was

created for its sheer intellectual delight on the one hand and for fighting wars on the other. Most of these applications require only present-day upper-primary-level mathematics.

Let us consider some examples of these applications:

EXAMPLE 1. Construction of altars This requires finding volume of altars of different shapes and finding the number of bricks for constructing these. This also requires all the formulae for areas and volumes studied in in the mensuration course.

EXAMPLE 2 Finding the radius of the Earth

Erosthenes, almost the first real scientist, solved this problem ?
The principle is illustrated below:

Let θ_1 and θ_2 be the angles between the vericals at A and B (ie the lines joining A and B to the centreoof the earth) and the direction of the pole star (ie direction of the axis of the earth)

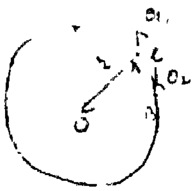

Figure 1

Then from figure I

$$l = r(\theta_2 - \theta_1)$$

$$r = \frac{l}{\theta_2 - \theta_1}$$

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so that knowing l, θ_1, θ_2, we can find r, the radius of the earth.

EXAMPLE 3 Finding the height of a tower

From figure 2

$$\frac{H}{h} = \frac{b}{a} \quad \text{or} \quad H = h\frac{b}{a}$$

We can easily measure a, h and b, so that we can find H

We do not need trigonometry. We need only properties of similar triangles

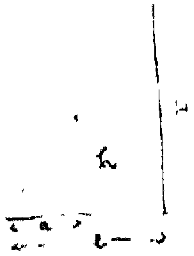


FIGURE 2

EXAMPLE 4 Finding the distance of the Moon

In figure 3 measure angles $\theta_1, \theta_2, \phi_1, \phi_2$,

Take the radius of the earth tobe 1.

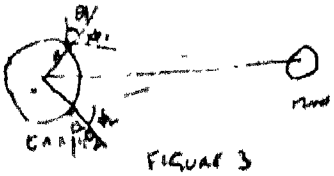


FIGURE 3

We can draw similar triangles and estimate the distance of the moon.

APPLICATIONS TO DAILY LIFE

EXAMPLE I COMPOUND INTEREST

The development of the formula

$$A = P(1 + r/100)^n$$

provides motivation for the learning of algebra, a generalisation of arithmetic. Its applications require the use of calculators. Thus let the amount of a savings certificate becomes double in 6 years, interest being payable half-yearly, we have to find the rate of interest. Then

$$200 = 100(1 + r/200)^{12}$$

$$\text{or} \quad 1 + r/200 = 2^{1/12} = 1.05946$$

$$\text{or} \quad r = 11.892$$

If the interest is payable quarterly, then we will get

$$200 = 100(1 + r/400)^{24}$$

$$\text{or } 1 + r/400 = 2^{1/24} = 1.02930$$

$$r = 11.72$$

If the rate of interest is 12% and the interest is payable quarterly then in 6 years, Rs 100 will become

$$100(1 + 3/100)^{24} = \text{Rs}203.2794$$

All these calculations are almost impossible without calculators and these examples represent real-life problems.

EXAMPLE 2 India's population today is 80 crores and is growing at the rate of 2.3 % per year. In how many years will it become double? What will be the population of India in the year 4000 A.D? How much land area will be available per person? If the population grows at the rate of 1% per year what will be the population in the year 4000 A.D.?

If the population doubles in n years, then

$$2 = (1.023)^n$$

This requires the knowledge of logarithms and is beyond the upper primary level. However the answer is 3.48 years .

This will enable us to calculate the population approximation

because in 2010 years there will be approximately 65 such doubling periods so that the population would be

$8\cdot 0\times 2^{65} = 80\times 3.7\times 10^{19}$ crores (This and all calculations below done with the help of calculator)

The exact value will be

$$80\times(1.023)^{2010} = 80\times 7\times 10^{19} \text{ crores}$$

The land area of India is 2.7 million square km.

Land available per person is

$$= \frac{2.7\times 10^{6}\times 10^{6}\times 10^{4}}{80\times 7\times 10^{19}\times 10^{7}} \text{ sq cm} = \frac{2.7\times 10^{16}}{56\times 10^{27}} = \frac{1}{2}\,\frac{1}{10^{12}} \text{ sq cm}$$

If the population grows at the rate of 1% per year, the population will be $80\times(1.01)^{2010} = 80\times 485245261$ cms

and the area per person will be

$$\frac{2.7\times 10^{16}}{4\times 10^{17}} = \frac{1}{15} \text{ sq cms}$$

Even if we cut down the birth rate to 1%, we shall have 1/45 sq cm per person in the year 4000 A.D.! We shall have to reduce the growth rate to zero percent if we have to survive in the world.

EXAMPLE 3 Volume of blood in the human body

Suppose a person takes 50 gms of sugar and after a few minutes the measure of sugar in a sample of 10 c c of blood is .1 gm, then by the simple rule of three ,the volume of blood comes out to be 5000 cc or 5 liters.

EXAMPLE 4 Surface Area of air sacs in the human body

The air we breathe passes through the trachea ,which divides it into main branches, the right and left bronchia, each of which divides into two ducts, each of which again divides into two more and this process of bifurcation continues 20-22 times . Each of the terminal ducts has above 300 alveolar sacs at the end. The diameter of each ~~of each~~ alveolar sac is 75-300 microns ,where a micron is equal to 10^{-6} meters.

The total number of ducts is between 2^{20} and 2^{21}

Now $2^{20} = (2^{10})^2 = (1024)^2 \simeq 10^6$

The total number of ducts is between one million and four millions and the total number of alveolar sacs is between 300 millions and 1200 millions Usually the number is taken to be 300 millions.

If we take the average radius of an alveolar sac as 100 microns, surface area is

$$4\ (10^{-4})^2 \text{ sq meters} \simeq 12 \times 10^{-8} \text{ sq meters} \simeq 10^{-7} \text{ sq meters}$$

The total surface area of all alveolar sacs would then be

$$300 \times 10^6 \times 10^{-7} \simeq 30 \text{ sq meters}$$

It has been estimated that the total surface area is about 70 sq meters The total surface area of the body of an average adult human being is 1.7 sq meters (how can you find this area?), so that if all alveolar sacs are [illegible] spread over the human body, they would cover the whole human body four timesover. If these are spread on the ground, they would cover a badminton court!

MOTIVATIONS FOR LEARNING SOME TOPICS UPPER PRIMARY LEVEL MATHEMATICS

(1) Square root (a) Find the length of the sides and the perimeter of a square whose area is given (b) solve the equation $x^2 = a$ $(a > 0)$ (c) solve the equation $ax^2 + bx + c = 0$

(11) Cube Root a. Find the length of the sides and perimeter and area of the faces of a cube whose volume is given.

(b) solve the equation $x^3 = a$

(c) solve the equation $ax^3 + bx^2 + cx + d$

(111) Fourth and higher roots (a) Finding the length of sides of hyper-cubes whose hyper-volume is given (b) solving higher degree equations.

(1v) Geometry and Mensuration (a) Studying properties of triangles, squares, rectangles, trapeziums, parallelograms, polygons, circles

(b) surface areas of fields, walls and houses, volumes of containers

(c) studying symmetry of figures

(v) Commercial Mathematics [illegible] of national [illegible] number of cars in Delhi 1 million and going up by 10% per year [illegible] the pollution caused by them c [illegible] the problems of national certificates units, insurance, environment mathematics, space mathematics; where do we stand in relation to other countries?

(vi) Statistics; statistical representation - of data, in terms of histograms, charts, bar diagrams etc. Celebrate statistical day every year.

MATHEMATICS AS A TOOL OF LOGICAL REASONING

One of the objects of teaching mathematics should be to teach the students the nature of mathematical thinking. This includex axiomatic, logical and deductive thinking, being precise, knowing the meaning of if and only if, necessary and sufficient conditions, analysing what is given and what has to be proved, finding which variables have to be retained and which variables have to be eliminated.

SUGGESTED READINGS

- SUGGESTED EXPERIMENTS IN SCHOOL MATHEMATICS (in three volumes) published by Arya Book Depot, Delhi. This gives 500 [illegible] active innovative ideas for discovery and pattern recognition approaches (teaching and learning
- INSIGHT INTO MATHEMATICAL MODELLING, INSA. This gives 100 applications of school mathematics in surveying the earth and the heavens, ecology, human physiology, physics. economics, operations research etc
- THE FASCINATING WORLD OF MATHEMATICAL SCIENCES (in 8 volumes) Mathematical Sciences Trust Society, Volumes 2 and 3 of this give fifty essays on the applications of mathematics
- EXCELLENCE IN TEACHING, For all those who want to become better teacher
- Mathematical Modelling, Wiley Eastern. This gives 100 applications of undergraduate mathematics
- Mathematical Models in Biology and Medicine, Affiliated East West Press

SOME QUESTIONS ASKED AT THE NCERT [illegible] TEACHERS FROM PROF J.N.KAPUR AND HIS ANSWERS TO THESE QUESTIONS

What is a set?

A set is a well-defined collection of numbers ie given any object, we should be able to say whether it belongs to the collection or not. Thus we have a set of natural numbers, but we do not have a set of good cricketers, because a good cricketer is not very well-defined.

Why do we study sets in mathematics?

Mathematics can be defined as a study of sets with structures. We study sets of natural numbers, integers, rational numbers, complex numbers, vectors, matrices etc and the structures are provided by commutative, associative, distributive laws etc. We also study sets of points and lines in geometry and the axioms and theorems provide the structures there.

But do we study these structures in school mathematics?

To some extent, we do so, when we study various laws of algebra and various theorems of geometry. However we do study the algebra of sets, where instead of addition and multiplication of numbers, we study union and intersection of sets.

What is the structure in the set of sets?

This structure is provided by laws like the following:

$A \cup A = A, \qquad A \cap A = A$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C$

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

Some of these laws are different from the laws of algebra of numbers.

What are the uses of study of sets?

(i) It shows that we can have an algebra different from algebra of numbers e.g here we have

$A \cup A = A$ but in algebra of numbers $a + a \neq a$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, but in algebra of numbers

$$a + (b \times c) \neq (a + b) \times (a + c)$$

(ii) It gives us the algebra of logic. We may consider the set of all true statements, then a statement is true if and only if it belongs to this set.

(iii) It gives us the algebra of switches. A switch can be open or closed and switches can be combined either in parallel or in series.

(iv) This gives the BOOLIAN algebra underlying computers

Why is the product of two negative numbers a positive number?

Since we need the commutative, associative and distributive laws to hold for all numbers, positive or negative. If the distributive law is to hold, then

$$a \times (b + c) = a \times b + a \times c$$

so that $a \times (b + (-b)) = a \times b + a \times (-b)$

but $b + (-b) = 0$ and $a \times 0 = 0$

so that $0 = a \times b + a \times (-b)$

$$\text{or } a \times b = -(a \times (-b))$$

$$\text{or } (-a) \times (-b) = -((-a) \times b), \quad a > 0,\ b > 0$$

$$\begin{aligned} \text{Thus} \quad (-3)(-4) &= -((-3) \times 4) \\ &= -((-3) + (-3) + (-3) + (-3)) \\ &= -(-(12)) \\ &= 12 \end{aligned}$$

Here we have used the fact that negative of a negative number is positive.

What is 'Vedic Mathematics'?

This is the name of a book written by Jagatgurum Shankaracharya Tirathankar Ji Maharaj.

What does the book contain?

It contains sixteen sutras giving high speed methods of arithmetical calculations.

Do these sutras always take less time than the usual methods?

No, Not always; but where they are applicable, they lead to a considerable saving of time.

Are these formulae given in the Vedas?

No, It was originally claimed that these sutras were given in an appendix to Atherva-Veda, but there is absolutely no evidence for this. In fact some of these formulae depend on post-Vedic mathematics and mathematics developed as late as fifteenth and sixteenth centuries. Swamiji was an M.A in mathematics and knew all about this. However great mathematicians like Aryabhatta and Bhaskaracharya did not talk about these and if these had occured in the Vedas, these mathematicians would have certainly discussed these.

Q. But it is said that all knowledge is contained in the Vedas and so Vedic mathematics must also be in the Vedas.

A. Why only Vedic mathematics; according to this, even mathematics yet to be created must be in the Vedas! All mathematics, all science and all techno-

logy are Vedic and therefore no book has a right to have the title Vedic mathematics.'

What is problem-solving in mathematics? Why is it being given so much importance?

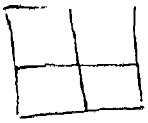
Problem-solving is the heart of mathematics. Mathematics is not computation. Doing mathematics is thinking mathematically. The more important thing is to know the technique of solving non-routine problems. According to Polya's scheme, we have to decide about the following:

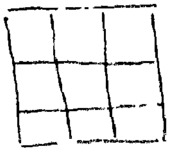
- What is given in the problem?
- What is to be proved in the problem?
- What is the best path from what is given to what has to be proved?
- Have you solved similar problems before?
- Can you solve a simpler problem first?

and so on.

Can you illustrate the strategy?

Consider the problem of finding the number of squares of all sizes on a chess board. We can try to count all of them, but this will take time and we can make mistakes. As such we try the simpler problem for 2x2 and 3x3 squares.





We find the number of squares is 1^2+2^2 and $1^2+2^2+3^2$ respectively, so that on a chess board we have

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2 = 204 \text{ squares}$$

Can you give more problems of the same type?

(a) Find the number of rectangles of all sizes on a chess-board.

(b) Find the number of squares on a 100x100 board.

(c) Find the number of rectangles on a 100x100 board.

(d) Find thenumber of cubes of all sizes in a nxnxn cube.

Can you give us some references?

You can read Polya's book on "How to solve it" and my book on "Enjoyment of mathematics for the millions" published by Arya Book Depot. This contains fifty such problems. My 4 volumes of "Fascinating World of Mathematical Sciences" and three volumes of Suggested Experiments in School Mathematics also contain many interesting and challenging problems Problems set in olympiads are also useful for this purpose.

NUMBER SYSTEMS

By

SHASHI PRABHA ARYA
Maitreyi College, Bapu Dham Complex, Chanakya Puri
New Delhi-110021

First of all, I would like to thank the organisers of this workshop for inviting me. When Ms. Surja Kumari approached me, she said you have to speak on Number Systems-but not the stuff given in the text-books, something new that they can enrich their teaching with, something that would enthuse- that would excite - that would activate the students. With that in mind, I would try to give you some scattered ideas which you can use in your class-room while developing your number systems as you go along from class VI to class VIII.

Infact, number system forms the basis of all mathematics. The branch of mathematics which deals with the properties of numbers is called Number Theory. The Prince of Mathematicians Gauss once said

"Mathematics is the queen of Sciences and the number theory is the queen of mathematics." The beauty of the theory lies in the fact that there are lots of unsolved famous problems which can be easily explained to the students and they can easily understand these. I shall try to describe some of these problems and you would find that these would surely generate a lot of interest in the students towards mathematics. It is very important that the students are not scared away from mathematics right at an early stage.

As you know, the students already have an idea of numbers. They are forced to learn about numbers and operations of numbers while they are still too young. What they are not exposed to is the beauty, the power, the very many uses of the number systems.

The early important developments based on numbers must be credited to Pythagoras and his students. To Pythagoreans, everything was number. The meaning they attached to numbers seem childish to us today. For them, one stood for reason, because it was through reason that one could arrive at one consistent body of doctrines. Two was identified with opinion since to every opinion there is an antiopinion. Three was identified with man(infact odd numbers were masculine, even numbers feminine) being the first masculine number, four was justice since it is the first number which is the product of two equals, five was identified with marriage being the union of two (the first feminine number) and three, seven represented health and eight represented friendship or love. The four fundamental operations of addition, subtraction, multiplication and division evolved gradually in the present form were referred to by Lewis Caroll as ambition, distraction, uglification and derision. By the way, so far as the Hindu contribution to the development of number systems is concerned, we would like to recall that the idea of using zero was conceived by the Hindus. It reached Europe through the Arabs. Also, the decimal system of notation of numbers is due to Hindus.

The books 7, 8 and 9 of Euclid's 'Elements' are devoted to number-theory. As you know, the 'Elements' is the oldest Greek

treatise which summarizes the whole of mathematics [illegible] the time of Euclid. It is in the form of 13 books. People have had so much fun from numbers. So many books on mathematical recreation deal with interesting puzzles [illegible] numbers. They not only reveal the [illegible] developments in number theory have [illegible] Before I describe some of the famous problems of classical number theory, let me give you methods of constructing magic-squares of order n. Magic squares have been a form of old amusement. In ancient times, magic squares of orders 3, 4, 5, 6, 7, 8, 9 were [illegible] with the seven astrological planets : Saturn, Jupiter, Mars, Sun, Venus, mercury and the Moon. Ramanujan's first [illegible] started with his work on magic squares.

Natural numbers can be classified into 3 categories : Odd, sigly even and doubly even. I shall tell you a rule to construct a magic square of any order n. Ofcourse, there is a mathematical theory behind the rules and those interested can refer to Rouse Ball's book for details. There is a story that in the court of the Russian queen Catherine the great, Euler was called upon to prove the existence of God. After writing down long complicated mathematical formulae which had no meaning, Euler wrote in the end 'and hence the God exists'. Since no one present understood a bit, everyone agreed that the God exists. It is not that you ~~accept~~ should accept anything in mathematics without understanding the theory underneath, but so far as the three rules for constructing the magic squares are concerned,

I am sure the students would enjoy even if they don t appreciate the theory behind.

Rule for constructing a magic square of order n where n is odd :

Suppose $n = 2m + 1$.

Start with the number 1 and place it in the middle column of the top row.

The successive numbers should now be placed in their natural order in a diagonal line which slopes upwards to the right, except that

(a) When the top row is reached, the next number is written in the bottom row as if it came immediately above the top row ;

(b) When the right-hand column is reached, the next number is written in the left-hand column as if it immediately succeeded the right-hand column ;

(c) When a cell which has been filled up already or when the top right hand square is reached, the path of the series drops to the row vertically below it and then continues to mount again.

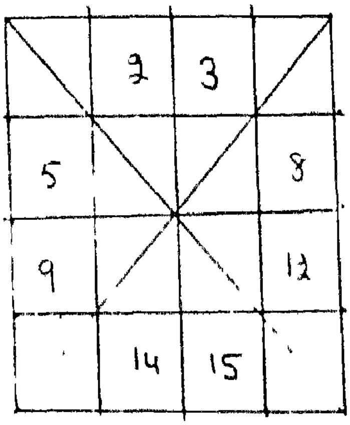
Rule for constructing magic squares of a doubly even order

Suppose n is a number of the form 4m.

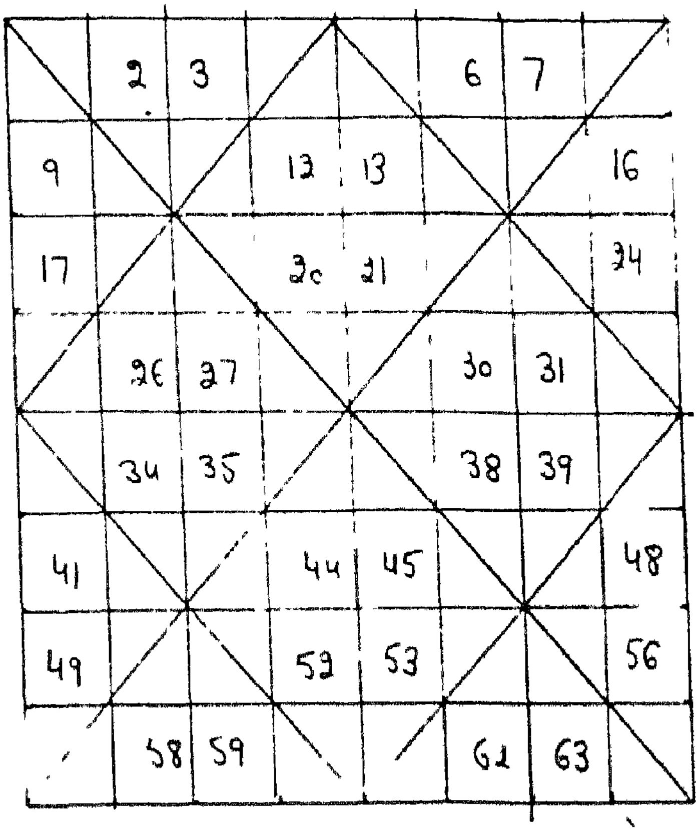
First write the numbers from 1 to n^2 in their natural order in rows of n.

Cross the numbers by the diagonals of every component block of 4^2 cells.

Now, replace the numbers in these diagonal cells by their complements.



n = 4



n = 8

Rule for constructing magic squares of a singly-even order:

Suppose $n = 2(2m+1)$. Divide the square in to four equal quarters A, B, C, D, in the order as shown.

A	C
D	B

Let $u = \frac{n}{2}$. Then u is odd.

In A, construct a magic squares with numbers 1 to u^2. Then, construct magic squares in B, C and D with numbers $u^2 + 1$ to $2u^2$, $2u^2 + 1$ to $3u^2$, $3u^2 + 1$ to $4u^2$.

Now, in the middle row of A, take the m cells next but one to the left-hand side, in each of the other rows take the m-cells nearest to the left-hand side and interchange the numbers in these cells with the numbers in the corresponding cells in D.

Next, interchange the numbers in the cells in each of the m-1 columns next to the right hand side of C with the numbers in the corresponding cells B.

Problems in Prime Number Theory

You are already familiar with prime numbers. Let us recall that a natural number $p > 1$ is said to be a prime number if its only divisors are ~~1, p.~~ 1 and p .

For instance, as you can check , first 10 primes are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

1 is not considered to be a prime.

A natural number $p > 1$ is said to be a composite number if it has divisors other than 1, p.

For instance, 6, 27, 35 are composite.

Every natural number greater than 1 is either prime or composite.

Let us also recall the following important theorem which you have already learnt :

The fundamental theorem of Arithmetic (also known as the unique factorization theorem):

Every natural number greater than 1 can be expressed uniquely as a product of powers of prime numbers.

In view of the above theorem, the prime numbers become very important, as it reduces the properties of numbers to the properties of prime numbers.

Several problems were raised in connection with primes. One of the major questions asked was

"How many primes are there ?"

One of the first significant contribution to the theory of prime numbers came from Euclid who answered the above question by proving that the number of primes was infinite. The proof appeared as proposition 20 in book IX of Euclid's 'Elements'. The proof is remarkable because of its simplicity. It is a beautiful illustration of excellence in mathematics. G.H. Hardy writes about this theorem.

'It is a 'simple' theorem, both in idea and in execution, but there is no doubt at all about its being a theorem of the highest class. It is as fresh and significant as when it was discovered - two thousand years have not written a wrinkle on it.'

Let us describe the proof as given by Euclid.

Suppose the set of primes $2, 3, 5, \ldots, p_n$ is a finite set.

Form the number

$$m = (2.3.5. \ldots\ldots p_n) + 1 \quad .$$

The number m must be either prime or composite. If m is prime, we have nothing to prove since in that case m is a prime not in the given set (it being greater than any of them). If m is composite, let q be one of its prime factors (this is guaranteed by the Fundamental theorem of Arithmetic). Now,

$q \notin \{2, 3, 5, \ldots\ldots, p_n\}$ because there is always a remainder of 1 when m is divided by each of these. Therefore the given set does not contain all the primes and so our supposition is wrong and the primes are not finite in number.

You may like to know that in 1837, Dirichlet proved that

every set of numbers of the form $an + b$, where a and b are relatively prime, contains an infinite number of primes. Thus, for instance, the set of integers of the form $3n + 1$ (3 and are relatively prime, that is, they have no common divisor other than 1), that is, the set $\{1, 4, 7, 10, 13. \ldots\}$ contains infinitely many primes. Similarly, the set $\{6n + 5 : n \in \mathbb{N}\} = \{5, 11, 17, 23 \ldots$ contains infinitely many primes.

In an attempt to find a simple algebraic formula that will produce only prime numbers, Fermat conjectured that $2^{2^n} + 1 (= F_n)$ was a prime for all positive integers n. F_n's are known as Fermat + primes. For $n = 1, 2, 3, 4$, we get respectively the primes 5, 17, 257 and 65537. But Euler (1732) showed that for $n = 5$, $2^{2^5} + 1 = 4294967297$ is $= 641 \times 6700417$ and so not a prime number. Infact it has not been proved that $2^{2^n} + 1$ is a prime for any value of n greater than 4. It is not a simple matter to verify it even for small values of n. For example, even for $n = 7$, $2^{2^n} + 1$ is greater than 34 followed by 37 zeroes.

F_n's which are not prime : (in order of discovery date) are F_{12}(1877), F_{23}(1878), F_6(1880), F_{36}(1886), F_9, F_{18}, F_{38}(1903), F_7(1905), F_{73}(1906), F_8(1909), F_{15}(1925), F_{10}(1952), F_{16}(1953). F_{17} is the smallest whose character is not known. There are at least 35 F's which have been proved to be composite. The largest is F_{1945} which has a factor $5.2^{1947} + 1$ which has 587 digits.

There are some formulae which give primes upto a cetain point. For instance, $n^2 - n + 41$ is prime for all $n < 41$. Likewise, $n^2 - 79n + 1601$ is prime for all $n < 80$. But no formula has yet been

established which would give a prime for all values of n.

Wilson's beautiful completely general theorem: If $n > 1$, then $(n - 1)! + 1$ is a multiple of n if and only if n is prime.

There are several other questions concerning primes. For example, how many pairs of primes differ by 1, how many differ by 2 (~~primetwins~~ prime twins), or by 3, or by 4 etc. Some of these are easy to answer. For instance, 2 and 3 are the only primes differing by 1, 2 and 5 are the only primes differing by 3. There are no primes differing by 7. But there remain several unsolved problems concerning primes.

Another question concerning primes that is asked is 'how far apart are different primes ?'

In this connection, one might like to note that none of the following 999 consecutive natural numbers is prime :

$$1000! + 2 , \quad 1000! + 3 , \quad 1000! + 4 , \ldots\ldots$$
$$- - - - - \quad 1000! + 1000 .$$

Can you see why these are not primes ? Also, are you wondering why we didnot start with $1000! + 1$? The reason is that $1000! + 1$ could well be prime. It is not settled till today.

In 1641, Fermat proved that every prime of the form $4n + 1$ can be written as the sum of two squares in a unique way. This reminds one of a famous story about Hardy and Ramanujan. Hardy had gone to see Ramanujan in hospital and he told Ramanujan 'I thought the number of my taxi was 1729. It seemed to me rather a dull number'. To this, Ramanujan replied 'No, Hardy! No, Hardy! It is a very

interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.' Can you write 1729 as the sum of two cubes in two different ways ? When Hardy asked Ramanujan whether he knew the answer to the corresponding problem for fourth powers, he replied that the first such number must be very large. Hardy found out later that Euler had solved the fourth-power problem already and the number was

$$635,318,657 = 158^4 + 59^4 = 134^4 + 133^4 .$$

One other problem concerning primes has been that of finding the largest prime. In 1965, the largest known prime was

$2^{11213} - 1$ which contains 3376 digits.

To-day, the largest known prime is $2^{216091}-1$, a number of 65 050 digit

These days with the help of computers, it is possible to examine very large numbers for their primeness. But still, there are many problems which remain unsolved and there are many which have been solved only in theory one remark is in order:

Suppose we want to examine whether a given number c is prime or not. One way would be to try to divide the number by all numbers less than it. However, since every non-prime will have a prime factor, it suffices to divide by all primes less than c. However still, in a factorization not both factors a, b can be $> \sqrt{c}$, for $a > \sqrt{c}$, $b > \sqrt{c} \Rightarrow ab > c$ which is absurd.

Thus, to find whether c has a divisor we need only examine whether any of the primes $\leq \sqrt{c}$ is a divisor.

Ex. $\sqrt{C} = 91 = 9'$--- . By trying 2, 3, 5, 7, we find 91 = 7 x 1

The most extensive table of primes giving all primes upto 10 million was published by D.N. Lehmer. There are exactly 664579 primes less than 10 million, or about 6 $\frac{1}{2}$%. More recently, D.H. Lehmer (the son of D.N. Lehmer) calculated total number of primes less that 10 billion. There are exactly 455,052,512 such primes or about 4 $\frac{1}{2}$ %,

Mersenne's numbers are as famous as Fermat primes. They are defined by

$$M_p = 2^p - 1 \text{ , where p is a prime.}$$

In 1644, Mersenne asserted that the only values of p below 257 for which $2^p - 1$ is a prime are

$$p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257 .$$

Unfortunately this was not true. In 1886, it was discovered that M_{61} is a prime. However, in 1903, it was proved that

$$M_{67} = 193707721 \times 761838257287 \text{ and so } M_{67} \text{ is not a}$$

prime.

Moreover Lehmer proved in 1931. that M_{257} was not a prime where as M_{89} and M_{107} were proved to be prime.

Today it is known that M_p is prime for

$p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213$ and

M_p is not prime for all other values of p < 12000.

Thus there are 30 Mersenne primes known today of which the largest happens to be [illegible]. ~~It was discovered by a~~

~~computer in the Mathematics Department of the University of Illinois who were so proud of it that they issued a stamp with the number.~~ It is still unknown whether the number of Mersenne primes is infinite or finite.

Mersenne primes are very much related with perfect numbers. A number is said to be perfect if it is the sum of its divisors other than itself. For example ,

$$1 + 2 + 3 = 6$$
$$1 + 2 + 4 + 7 + 14 = 28$$

and so 6 and 28 are perfect numbers. The smallest perfect number is 6. It has been proved that if $2^p - 1$ is a prime, then $2^{p-1}(2^p - 1)$ is a perfect number. Thus corresponding to every Mersenne prime, there is a perfect number. We mentioned that so far 23 Mersenne primes are known, so we know of 23 perfect numbers. Also, all the perfect numbers of the above form are even and it is possible to show that if a perfect number is even, it must be of this form. We may wonder :

Are there any odd perfect numbers ? As at present the answer is : We don't know. It would be quite something if one can come out with an odd perfect number. In 1968, it was shown that an odd perfect number must have atleast 36 digits.

Closely related is another class of numbers-namely amicable numbers or friendly numbers.

Two numbers are called amicable if each equals the sum of the aliquot divisors of the other. The oldest known such pair is

220 and 284.

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

and $$1 + 2 + 4 + 71 + 142 = 220.$$

The pair (220, 284) also happens to be the smallest pair of amicable numbers.

Again attempts were made from time to time to discover formulae which will produce amicable numbers. Today there are 390 known pairs of amicable numbers. A complete list giving all these 390 pairs can be found in Scripta Mathematica (1946) in the article on Amicable numbers by E.B. Escott who alone found 233 pairs out of these 390 in 1934. Infact there is one more pair (12285, 14595) of amicable numbers besides these 390 which was discovered by B.H. Brown in 1939. It seems that after brown, no new pair has come to light.

2. GOLDBACH'S CONJECTURE

Christian Goldbach was a German teacher of Mathematics. In 1742, he wrote to Euler that he conjectures :

"Every even integer (>4) can be written as the sum of two odd primes."

The above conjecture of Goldbach has not been settled till today. The conjecture is easy to verify for the first few even integers. For instance,

$$6 = 3 + 3$$
$$8 = 3 + 5$$
$$10 = 5 + 5$$
$$12 = 5 + 7$$

14 = 7 + 7

16 = 5 + 11

18 = 5 + 13

20 = 3 + 17

22 = 11 + 11

[illegible] + 1 [illegible]

[illegible] the conjecture must be true. Indeed, [illegible] conjecture [illegible] been verified for all even integers [illegible] 1 [illegible]. But a general proof, or a counter-example has [illegible]. Ofcourse, there are partial results. [illegible] even integer can be written as the sum of [illegible]. There are results which come very close to solving [illegible], the Russian mathematician Vinogradov [illegible] solved it. He has proved that

'Almost all integers are the sum of two primes.'

Here, 'almost all' means that if A(n) denotes the number of even integers $\leq n$ which can not be represented as the sum of two primes, then the ratio $\frac{A(n)}{n}$ is very near 0 as n becomes very large.

The mathematician Landau puts it like this

"The Goldbach conjecture is false for almost 0% of all even integers ; this almost 0% does not exclude, ofcourse, the possibility that there are infinitely many exceptions."

3 FERMAT'S LAST THEOREM

Consider the equation

$$x^2 + y^2 = z^2 .$$

There are several triples (x, y, z), which are natural numbers, which satisfy the above equation. Such triples are known as Pythagorean triples. For instance, (3, 4, 5), (5, 12, 13) are solutions of the above equations and so these triples are Pythagorean triples.

Infact, there are formulae which would give you several Pythagorean triples. For instance, for any natural numbers, m, n, the triple $(m^2 - n^2, 2mn, m^2 + n^2)$ is a Pythagorean triple, m, n are relatively prime $m > n$, exactly one is even ,other is odd. Likewise, if n is any odd number (> 1), then $\left(n, \frac{n^2 - 1}{2}, \frac{n^2 + 1}{2}\right)$ is a Pythagorean triple.

The situation is not that rosy when one attempts to find an integral solution of

$$x^n + y^n = z^n \qquad \text{for } n > 2 .$$

The French mathematician Pierre de Fermat made a conjecture that there are no three natural numbers x, y, z which can satisfy this equation if $n > 2$.

This result Fermat wrote in the margin of his copy of the Latin translation of Diophantus Arithmetica. The note reads as follows:

'To divide a cube into two cubes, a fourth power, or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.

Inspite of Fermat's firm declaration, the result remains unproved till today. Thus, it cannot be called a 'theorem', nor

it is the 'last' work of Fermat. A proof for n = 4 was found in Fermat's work elsewhere. For n = 3, it was proved by Euler a century later in 1770. Another century later, Legendre and Dirichlet proved it independently for n = 5 in 1825. In 1839, Lame proved it for n = 7.

In 1843, the German mathematician Kummer submitted a proof of Fermat's theorem to Dirichlet who found an error in the proof. Based on the ideas developed by Kummer, several special cases have been proved. For instance, it has been proved that Fermat's theorem is true for all $n < 4003$ and for many other particular values of n. But the general theorem has not been settled.

In 1908, the German mathematician P. Wolfskehl bequeathed 100,000 marks to the Academy of Science at Gottingen as a prize for the first complete proof of the theorem. This motivated thousand of amatuers to attempt a solution. Fermat's theorem has gained the reputation that it is a mathematical problem for which the greatest number of incorrect proofs have been published. There appeared 1000 false proofs between 1908 and 1912. Ofcourse, several eminent mathematicians have worked on the problem and many partial results have been obtained. For instance, the problem has been reduced to finding an integral solution of

$$x^p + y^p = z^p$$

where p is an odd prime. Whether even Fermat had himself found a solution or not might never be known.

The Number System and Conceptual Errors
in Mathematics.

Dr. Mukum Singh
D.M.S., N.C.E.R.T.

The Natural or Counting Numbers

Consider the set of natural numbers : $N = \{1, 2, 3, 4 \ldots.\}$ If we add any two of the numbers, the number which is their sum will also be in the set N. Thus given any pair of natural numbers as a and b, there exists one and only one natural number called the sum of a and b. This sum is indicated as a + b.

Now consider a subset of N such as the set of all one digit natural numbers : $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. (A set M is called a subset of N if every element of M is also an element of N). It may be seen that the sum of two numbers such as 3 and 5 will be another number in the set. However, it is possible to add two numbers, or elements, such as 6 and 8 to obtain a number not in the set. We say, then that the set of all one digit natural numbers is not closed under the operation of addition. The set of all even natural numbers is closed under the operation of addition, since the sum of any two even numbers is another even number. The set of all odd natural numbers is not closed for addition, since the sum of two odd natural numbers is an even number.

It may seem unnecessary to call attention to the fact that we obtain the same sum when we 'add up' a column as when we 'add down' the column. But this illustrates the principle of the commutative law for addition of

natural numbers. In general it means that if a and b are any two natural numbers, then $a + b = b + a$.

The associative law for addition means that if a, b and c are any three natural numbers, then $a + (b + c) = (a + b) + c$. To be specific $(2 + 3) + 4 = 2 + (3 + 4)$, or $5 + 4 = 2 + 7$, since each sum is 9.

The set of natural numbers is also closed under the operation of multiplication, for if we multiply any two natural numbers, we obtain another natural number. Thus given any two natural numbers a and b, there exists one and only one natural number called the product of a and b. This product is indicated as $a \cdot b$ or ab.

The commutative law also applies to multiplication of natural numbers. It means that if a and b are any two natural numbers, then $a \cdot b = b \cdot a$. To be specific $3 \cdot 4 = 4 \cdot 3$: which actually means that $4 + 4 + 4 = 3 + 3 + 3 + 3$.

Similarly, the associative law holds for multiplication of natural numbers, since if a, b and c are any three natural numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. To be specific: $(3 \cdot 4) \cdot 5$ which is $(12) \cdot 5 = 60$ is the same as $3 \cdot (4 \cdot 5)$ which is $3 \cdot (20) = 60$.

We may generalize further and say that the sum of any number of natural numbers is a natural number; we may add them in any order and group them in any way we please.

Also, the product of any number of natural numbers; we may multiply them in any order and group them in any way we please.

The distributive law of multiplication over addition means that $a.(b + c) = (a.b) + (a.c)$ and applies to the set of natural numbers.

The Integers. Do, we have closure for natural numbers under the operation of subtraction ? First let us define subtraction as the inverse of addition, meaning that if a and b are natural numbers, then there exists a unique number x such that the difference between a and b in that order means that $b + x = a$ or $a - b = x$. If $a > b$, then we see that x is a natural number; but suppose $a = b$ or $a < b$. In order to have closure for subtraction, the number system must be extended to include zero and the positive and negative numbers.

First, let us define zero, whose symbol is 0, as the number which has the property $a + 0 = a$ or $a - a = 0$. The set of natural numbers now become $N = \{0, 1, 2, 3, \ldots\}$. For any number b of this set, $b.0 = 0 + 0 + \ldots + 0$ where there are b zeros in the sum. Hence $b.0 = 0$. Also $0.0 = 0$. If a and b are any two natural numbers and $ab = 0$, then either a or b or both must equal to zero.

Let us define division as the inverse of multiplication. That is to divide a number a by a number b means to find a number x such that $b.x = a$. Then $a \div b = x$. Since $0 \div b$ means a number x such that $b.x = 0$, x must be zero if b is not zero. That is $0 \div b = 0$ if $b \neq 0$.

Does $a \div 0$, where $a \neq 0$, have a meaning? If so, it is a number x such that $0.x = a$. But since zero times any number is zero, this would lead to a contradiction, and

We must conclude that $a \div 0$ has no meaning. Also, if $0 \div 0$ means anything, it would mean a number x such that $0 . x = 0$. But then x could be any number so that $0 \div 0$ is meaningless. Therefore, we must have the rule that division by zero never permitted, and that $a \div 0$ is not a number.

To give meaning to $0 - a$ and $a - b$ where $a < b$. We must introduce positive and negative numbers. We shall call this new set of numbers, the set of new integers : $I = \{ \ldots\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ldots\ldots \}$. With these new numbers we have closure not only under the operations of addition and multiplication, but also under the operation of subtraction. Also the commutative, associative, and distributive laws still hold.

The Rationals. Before we examine the operation of division for clousre, let us restate the meaning of division. To divide a number a by a number b means to find a number x such that $b . x = a$. Then $a \div b = x$. In the examples $6 \div 2$; $(-6) \div 3$; $0 \div 5$ and $0 \div (-1)$, the quotients are also elements, or numbers, in the set of integers. Since $a \div 0$ or $0 \div 0$ has already been explained as having no meaning, we will need not consider such examples as $5 \div 0$ and $0 \div 0$. But is there a number x such that $-7 \div 2 = x$, meaning $2 . x = -7$? We see immediately that the set of integers $I = \{ \ldots\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ldots\ldots \}$ is not closed under division.

Since $-7 \div 2$ or $2 \div 5$ are not numbers in the set of integers, it is now necessary to introduce a new kind of number so that $a \div b$ will have meaning for all cases (except for $b = 0$). This new number is called a fraction and the symbol for it is a/b where a and b are natural

numbers. This would, however, restrict our new numbers to positive fractions. In order to include negative fractions, we must remove the restriction that a and b are natural numbers and say that they are signed integers. Then if our laws of operations for signed numbers are to hold, we must agree that if a and b are natural numbers, $\frac{-a}{+b} = \frac{+a}{-b} = -\frac{a}{b}$ and $\frac{-a}{-b} = \frac{+a}{+b} = +\frac{a}{b} = \frac{a}{b}$

This new system of numbers consists of all numbers of the form p/q where p and q are integers, q ≠ 0. This new system is called the rational number system. The system may be displayed as follows (the first row can be thought of as having denominator 1):

...... -4, -3, -2, -1, 0, 1, 2, 3, 4, ...
...... -4/2, -3/2, -2/2, -1/2, 0/2, 1/2, 2/2, 3/2, ...
...... -4/3, -3/3, -2/3, -1/3, 0/3, 1/3, 2/3, 3/3, ...
....
....

The rules which have been set up for operating with fractions can be extended to include all rational numbers and it can be shown that the set of rationals is closed under the operations of addition, multiplication, subtraction and also division. Hence, these four operations are called the rational operations.

The Real Numbers. Let us now consider a new kind of operation. Is it always possible to find a number such that the number multiplied by itself gives another number in the set of rationals? For example, the number 5 multiplied by itself gives the number 25; also, -5 multiplied by itself gives the number 25. Now what number multiplied

by itself gives 2 ? It can be proved that there is no rational number x such that $x . x = 2$ or $x^2 = 2$ by showing that x cannot be written as the quotient of two integers. This gives rise to a new kind of number called an irrational number and which is defined as a number which can not be written as the quotient of two integers. An algebraic irrational can be obtained as roots of equations of the form $P(x) = 0$, where $P(x)$ is a polynomial with rational coefficients. A transcendental number is an irrational, such as π which is not a root of any equation of the form $P(x) = 0$.

The real number system is now composed of the positive and negative rational numbers, the positive and negative irrational numbers, and zero. This system is closed under the operations of addition, multiplication, subtraction, and division by non-zero. The element of this system also follows the fundamental laws. If a, b, and c are real numbers, then : $a + b = b + a$; $a.b = b.a$ (commutative)
$a + (b + c) = (a + b) + c$; $a.(b.c) = (a.b) .c$ (Associative)
$a.(b + c) = a.b + a.c$ (Distributive).

Any system satisfying these conditions with two operations + and . , and in addition has a 'zero' element such that $x + 0 = x$ for any x in the system and has a 'unit' element such that $x.1 = x$ for any x in the system is called a field.

The Complex Numbers : The irrational number was invented in order to have a number whose square was either positive or zero. Is there a number whose square is negative ? We can say there is no real number but we can

invent a new number. Let us define this new number as a number whose square is negative and call it a pure imaginary number. We may denote the number whose square is −1 by i so that $i^2 = -1$. If a is any real number, then $(ai)^2 = a^2i^2 = -a^2$. We call i the imaginary unit. The word imaginary is misleading since actually the imaginary number is very real to an electrical engineer. Since the letter i is a symbol for electrical current, the physicist and electrical engineer use the letter j for $\sqrt{-1}$.

The system of numbers made up of real numbers and pure imaginary numbers is not closed for addition since a + b i is neither a pure imaginary number nor a real number. This new kind of number, called a complex number and defined as a number of the form a + b i where a and b are real numbers, gives us the complex number system. If a = 0, we have a pure imaginary number; if b = 0, we have a real number. The algebra of complex numbers is the same as for real numbers except that if we obtain i^2 we replace it by −1.

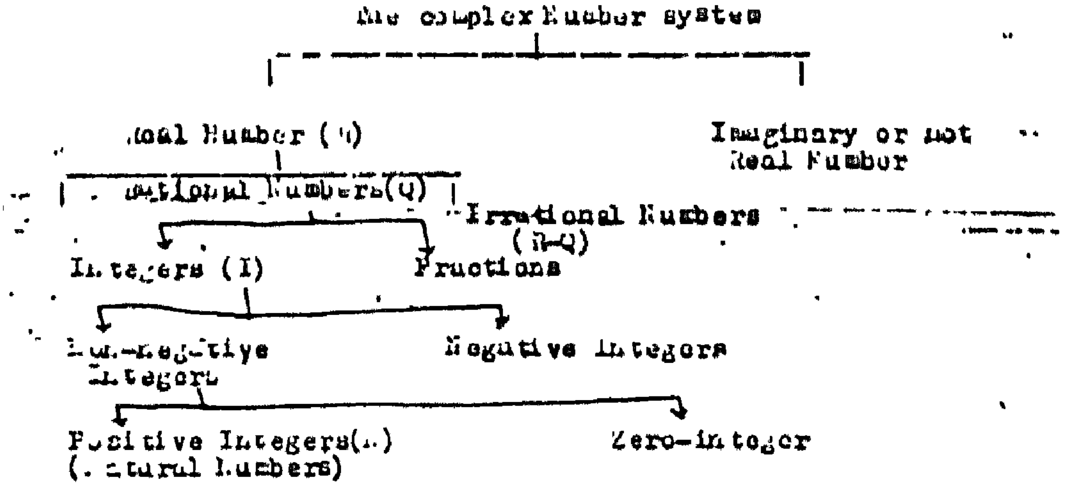
A Diagram of the Complex Number

It is important that the pupil sees the organization of the number system as it is developed. He should be made to realize that each new number was 'invented' because there was a need for it. For instance, one can imagine that there was a time when early man had no particular need for natural numbers. He had nothing to count. However, as his possessions increased, he like modern man wanted to talk to his neighbour about how much and how many he had, so 'counting numbers' or integers were invented. As civilization developed, life became more complex, making

the invention of still more numbers a necessity.

Beginning with the kindergarten programme and continuing through the last year of high school, the pupil should always be aware that he is building a system of numbers. By the time the pupil completes the study of complex numbers, he is ready to recognize the full impact of the principle of 'closure' on the development of the complex number.

The following diagram is helpful in showing the pupil the over-all picture. It can be used as a 'growing' bulletin board by adding the name of the number field as it is invented.



Graphing

The concept of 'coordinates' should be developed with order on a line. This can be done with a yardstick. A number is associated with each point (say the centimeter divisions) beginning with zero and continuing to include 36. Each of these is called the coordinate of a point. In other words, 2 gives the position of a point on the

[illegible]. Similarly, a line which contains only integers with each being associated with a point begins the development of the real axis.

Choose the set of positive integers $S = \{1, 2, 3, 4, 5, 6\}$. As graphically represented, - the 'solution set' of $x + 3 = 7$ is the point that has the coordinate (4) located on the line. See Fig. 1.

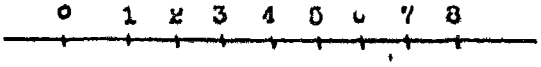


Fig. 1.

Now consider the graphical representation of $x + 3 < 7$. In the set of positive integers, the solution set consists of the points 1, 2, and 3 or $S = \{1, 2, 3,\}$ See Fig. 1.

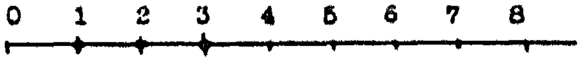


Fig. 2.

As the need arises for new numbers, the line axis is filled in with fractions and irrational numbers, both positive and negative. The real axis appears as illustrated in Fig. 3.

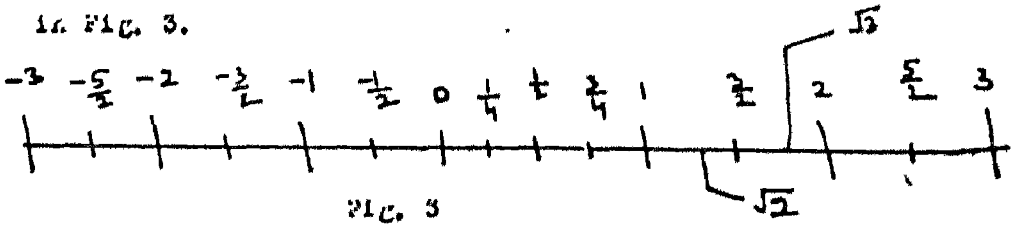


Fig. 3

In the set of positive real numbers, find the solution set of points for $x + 3 < 7$. This will be all the positive real numbers from 0 to 4 (but not inclusive) as indicated by that portion of the line axis enclosed with the brackets in Fig. 4. It means that the solution set will be a set of points for which $x > 0$ and $x < 4$, or $0 < x < 4$.

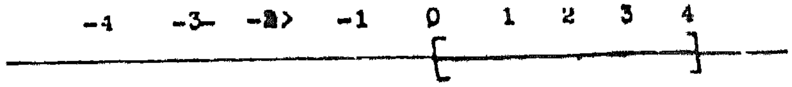


Fig. 4

Relationship with two variables become the next consideration. Consider the realation $y = x + 4$. Let x and y be selected from the set of positive integers. Now for every replacement of x with a positive integer, there is a possible replacement for y which will make the relation, $y = x + 4$, true. In Fig. 5, (1,5), (2,6), (3,7), etc is called a 'set of ordered' pairs'. This terminology seems quite natural, for the relation $y = x + 4$ actually orders these pairs. These pairs are first represented as 'lattice' points as shown in Fig. 6. They are infinite in number within the previously selected set of positive integers.

x	y
1	5
2	6
3	7
4	8
5	9
6	10
.	.
.	.
.	.

Fig. 5

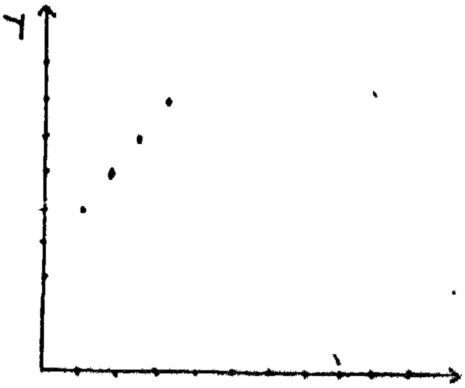


Fig. 6

Suppose x is selected from a set including only the positive integers from 0 to 4, ($0 < x < 4$). The solution set is now finite and includes : $S = \{(1,5),(2,6),(3,7)\}$ and the graph includes only the three lattice points indicated in Fig. 7.

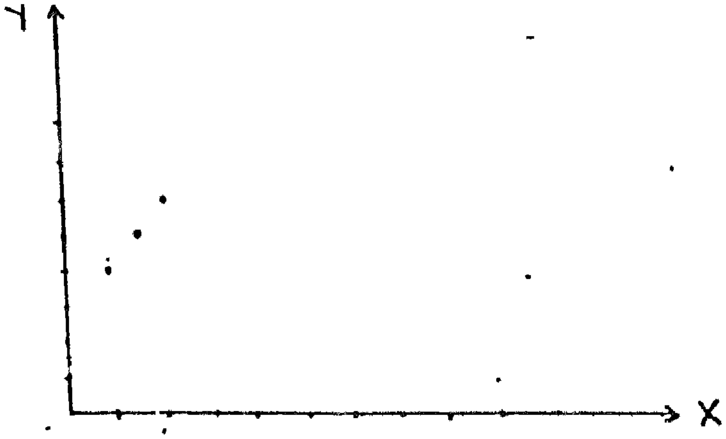


Fig. 7.

If the set from which x and y are selected include all the real numbers, the set of ordered pairs is infinite and the graph becomes continuous as indicated in Fig. 8.

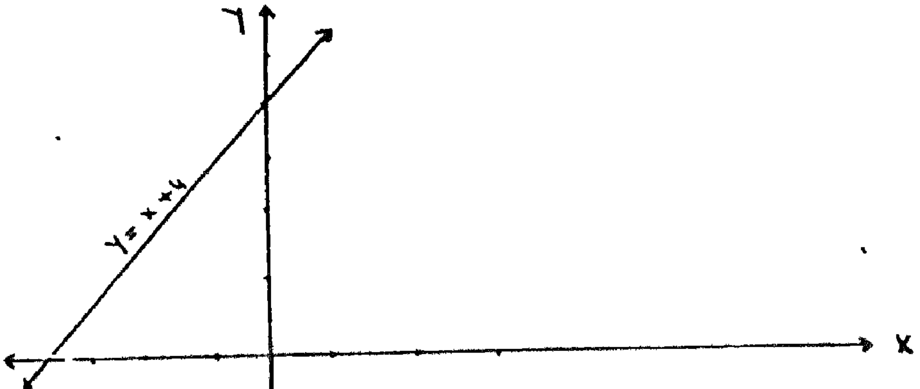


Fig. 8.

After the pupils have a thorough knowledge of relations which are equal, they should be asked to consider inequalities, sometimes referred to as inequations.

Consider the relation, $y < x + 4$ with x and y selected from the set of all positive integers. The solution set containing only positive integers will be infinite in number, examples being, (1,2), (1,4), (6,7) etc. The graph of these pairs of numbers, as shown in Fig. 9, becomes a set of lattice points which does not include the number pairs (1,5), 2,6), etc., of the $y = x + 4$ relation nor any points above these pairs.

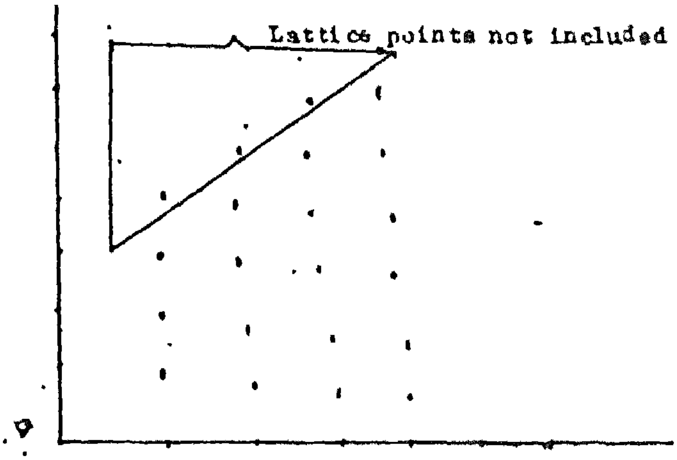


Fig. 9

If the set for x and y is increased to include all the real numbers, the set of points become a 'region' of the plane. The directed lines in Fig. 10 indicate this region.

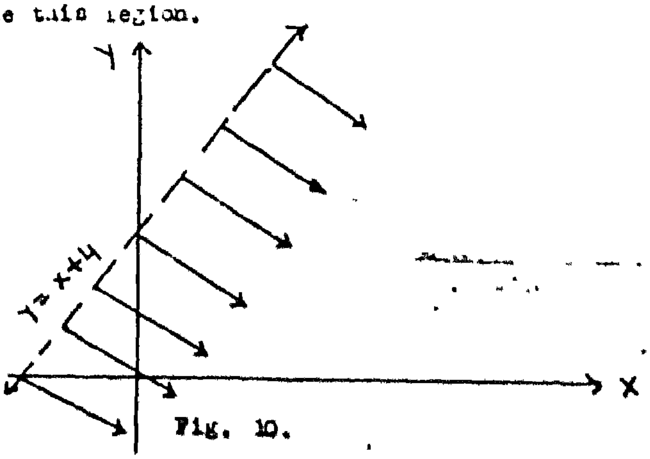
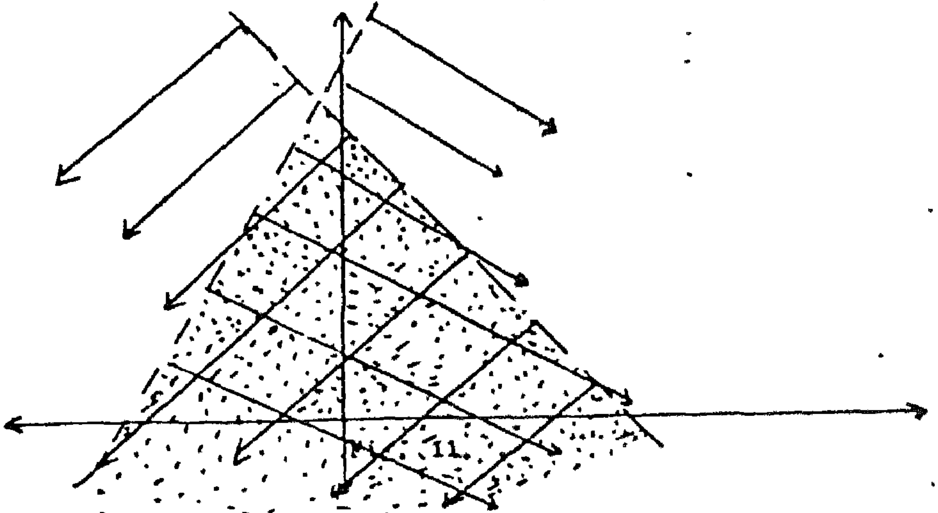


Fig. 10.

Many pupils will enjoy graphing a system of

inequalities. Propose the problem of locating the region which satisfies these two conditions : $x+y < 5$ and $2x-y < -7$. The shaded region shown in Fig. 11 constitutes the solution. This can be extended to apply to a system of 3 or more inequalities. Efficiency experts in industry utilize this method in 'linear programming'.



Conceptual Errors in Mathematics

Some conceptual errors in mathematics are very dangerous for students and their future careers. It has been observed that many authors make these errors in writing their textbooks, and the teachers and students become victims of these errors. Unfortunately, most teachers cannot detect these errors. All such conceptual mistakes in mathematics should be discussed in detail with teachers and students. Some such conceptual mistakes are given below.

1. The concept of 'inequation' and inequality' are considered to be synonymous by many authors. This is mathematically wrong. The difference between these two concepts is analogous to that between 'equation' and identity in mathematics. An inequation is a statement

ion is not true for all values of the variable(s) involved. But an inequality is a statement which is true for all values of the variables involved. Thus $x + y \geq 1$ is an inequation but $(x+y)^2 > x^2 + xy + y^2$ is an inequality, since it is true for all values of x and y of the same. We solve an inequation but prove an inequality.

2. The expression $\frac{x^2-9}{x+a}$ and $x-a$ (a being constant) are considered to be the same expression by many. This is also not true. For $\frac{x^2-a}{x+a}$ is not defined for $x = -a$ whereas $x-a$ is defined for $x = -a$.

3. Many teachers very often write $\sqrt{(x-y)^2} = x - y$ or $\sqrt{16} = \pm 4$, which is a wrong concept. In fact, the symbol '$\sqrt{\ }$' means the positive square root and not simply the square root. Hence $\sqrt{16} = 4$ (and not ± 4) and $\sqrt{(x-y)^2} = |x-y|$ and not $\sqrt{(x-y)^2} = x-y$ or $y-x$ or $\pm (x-y)$. But the square root of 16 which is written as $\pm$ 16, is definitely equal to $\pm$ 4.

4. A circle in geometry is thought to be a region having some area. But this concept is wrong. It is simply a curve (or a curved line) traced out by a point moving on a two dimensional space (plane) according to a certain law. It encloses a certain region. By the statement area of a circle, we mean the area of the region bounded by the circle and not the area of the curve which has no meaning.

5. The definition of two parallel lines as the lines which never meet is not perfectly correct. Two lines in space may never meet, but they still not be parallel. The point to be emphasised in the definition of parallel lines is that they must be coplanar. So the correct

definition of parallel lines is as follows : Two lines are said to be parallel if they are coplanar and non-intersecting.

6. Certain mathematical formulas are written in textbook without any mention of some restrictions. The violation of which may lead one to get fallacious results for which a teacher or a student may not be able to give a correct explanation, because of lack of complete information on the restrictions involved in the formula. These formulas are, therefore, incomplete, and hence not correct. It is the duty of teachers to point out these restrictions to the students. For example, the famous formula of _index law_, $a^m b^m = (ab)^m$ given in any standard textbook on algebra without mentioning any possible restrictions on the values of a, b and m. Now, we give the solution of a problem in two different ways : In one of the solutions, we use the above formula. As a result we shall get two contradictory answers. Unless a student or a teacher knows the restrictions on the values of a, b and m in the above formula he/she can not explain which of the two methods of solution is correct.

$$\sqrt{-16}\,\sqrt{-4} = (-16)^{1/2}\,(-4)^{1/2} = \{(-16) \times (-4)\}^{1/2}$$

$$= (64)^{1/2} = \sqrt{64} = 8$$

$$\text{also } \sqrt{-16}\,\sqrt{-4} = 4\sqrt{-1} \times 2\sqrt{-1} = 8 \times \sqrt{-1} \times \sqrt{-1}$$

$$= -8$$

Hence, in one case our answer is 8 and in another case it is -8. Now the question is, which one is correct and why ?

In fact, the formula $a^m b^m = (ab)^m$ is true in the following cases :

i) If m is an integer, then the above formula is true for all non-zero values of a, b.

ii) If m is a fraction of the type p/q where p and q are integers and the fraction is in the lowest term, then the formula is true for all non-zero values of a, b.

iii) If m is fraction of the type $p/_{2q}$ where p, q are integers, q > 0 and the fraction is in the lowest term, then the formula is true only when at least one of a, b is positive. When both a, b are negative, the formula does not hold good.

In the present example, since both −16 and −4 are negative the use of the above formula in the first method of solution is wrong and the second solution i.e. −8 is the correct solution.

These are only a limited number of examples of conceptual errors in Mathematics. There are many such examples where conceptual errors are made by authors. Teachers should try to find out as many such examples as possible and discuss them in a group for the improvement of Mathematics teaching.

ACTIVITIES FOR WORK ROOM IN COMMERCIAL MATHEMATICS

- SUNDER LAL (AGRA)

a) The purpose of this activity is to make the student understand the working of a bank. A person working in a nearby bank may be invited to come and describe the working of his branch to the students. Students then may be taken to that branch where they can have first hand information about its functioning.

b) The student should be taught how to deal with bank. Teacher (with the help of some bank officials, if necessary) should make the student aware of various types of accounts, forms, cheques, loan applications etc. For this purpose, blank forms can be produced from banks. These forms may also be got cyclostyled in the school. Students be then taught how to fill these forms correctly, how to fill pay-in-slip, how to fill a withdrawal form, how to write an order cheque, bearer chaque or a crossed cheque, how to fill a particular loan application form, how to get a draft prepared etc. with the help of bank officials, these forms must be checked as is done in banks. Various entries must be checked and corrected. Students may be given sufficient practice in this exercise

c) Students may be assigned small projects dealing with banks. They may be assigned various heads like fixed deposits, deposits in savings bank accounts, loan to agriculture loan to public sector, loan to weaker sections of the society, loan distributed in loan melas etc. They may be assigned different time periods like first quarter of a financial year or later half a particular calender year. They may then be asked to collect data for the assigned periods available in the bank under the assigned heads.

d) We enjoy lots of advertisements on T.V. These ads are about vari things being produced in the country. These products are being produced by various industrial houses. Some of these are in private sector and some in public sector. Students may be asked to collect these ads and tabulate the result as follows:

Product	Manufacturer	Sector(Public or Private)
1.		
2.		

e) A project similar to the one discussed in (d) may be done for the products advertised in news papers and radio.

f) LIC is also a deposit taking and loan giving agency. Projects similar to the one discussed in (c) may taken for different schemes of insurance.

g) Most of the newspapers give current rates in principal markets like bullion, oils, cotton, sugar, metal, spices, etc. Students may be asked to collect data for various items in different markets over a certain period. This data may be used to study variations, percentage of price rise or fall etc.

h) From the newspapers students may collect data regarding the trends in share markets. They may study market value of shares of a certain company over a certain period, and calculated profit gained or loss incurred if the money were invested in these shares.

i) Information regarding dividend declared by different companies may also be collected from newspapers. A comparative study of the dividends declared by different companies may be studied.

j) The following game may be used to familiarise the students the use of various tables.

Number of players: Two(A and B)

Material Needed: Two dice of different colours, some fake currency notes and coins, a table showing the amount of Rs.1 at rates 1%, 2%,......6% for the periods 1 year, 2 years, ... 6 years.

How to Play: Divide the money between the players equally. Name the two dice as a rate die$^{(R)}$ and time die$^{(T)}$. Player A puts some money, say x, at stake and asks for a number (from 1 to 6) on one of the dice. Suppose he asks for 3 on die R. Player B throws the two dice simultaneously. There are three possibilities. 3 appears on R. In this case B will give to A the amount of the principal x at the rate shown on the die R(3 in the present case) for the period shown on the die T. 3 appears on T. In this case A will give to B the interest on x at the rate shown on R for the period shown on T(3 in the present case). If 3 does not appear on any of the two dice. A loses the sum x to B. In the case of first possibility A gets the next chance of throwing the dice. In the other two cases B continues to throw.

The game may be played as long as the two players agree to continue. It may also end when one of the player has exhausted his money. Otherwise the two players count their money. The one with larger sum is the winner.

In the ab-ove game the amount is calculated as follows:

Find the figure in the table which lies in the row headed by the number shown on the die T and the column headed by the number on the die R. This figure is then multiplied by X to calculate the amount required under the first possibility. To calculate the interest required the case of second possibility we first compute the amount as in the case of first possibility and then subatract x from it.

Sevdral variations of this game can be thought of by changing the table and modifying the playing rules, if necessary.

AN ACTIVITY-CUM-RECREATIONAL APPROACH TO GEOMETRY

- ASHA RANI SINGAL
MEERUT UNIVERSITY
MEERUT.

Once upon a time, there was a philosopher named Plato who had written on the doors of his Academy: "Let no one ignorant of Geometry enter here!" Perhaps you are wondering what did a philosopher have to do with geometry! Well, Plato was a Greek philosopher who lived around 300 B.C. At that time, anything that involved a deep thinking process, fell to the lot of pilosophers(thinkers). Thus all the fine arts, science, astronomy and mathematics were studied by the philosophers and there was no separate category of men known as the mathematicians. This riddle solved, are you wondering as to why Plato considered Geometry to be so important that he did not want to set eyes on anybody who had no knowledge of this discipline? Now Plato, and other Greek philosophers before and after his time, had some very solid reasons of their own to study Geometry. To understand their motivation and desire to study this subject we shall step back about 2900 years in time.

Two thousand years before the Greek philosophers like Plato Egyptians and Babylonians knew several basic concepts about geometry and made practical use of the same. Egyptians used the subject mainly for the purpose of measuring the land and calculating the taxes there from. In fact, the word geometry is composed of geo and metry where 'geo' means Earth and 'metry' relates to measuring. Thus the literal meaning of the word geometry is land - measurement. Egyptians used geometric concepts in building their famous pyramids and things like that too. Babylonians made use of geometry basically in the study of astronomy. These ancient people seem to have used geometry on the basis of experiments and testing without worrying as to why a certain method produces a certain result. For example, let us consider the way an Egyptian might have gone about building a right angle. Special persons known as harpedonaptae(rope-stretechers) were employed to do the job. They had a rope which was knotted at distances in the ratio 3 : 4 : 5 say (Figure 1). The knotted

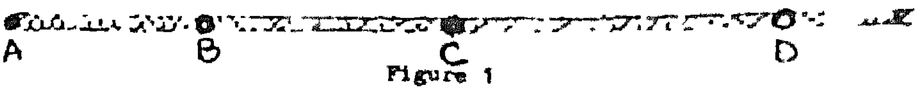


Figure 1

points A and B of the rope would be held at two points so that the portion AB is taut. A person would then move in a direction somewhat perpendicular to AB till the portion BC is

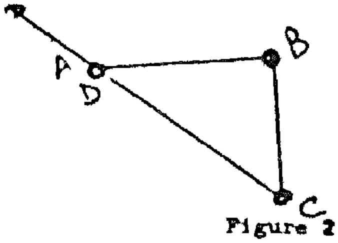


Figure 2

taut(Figure 2). Holding the knot C here, one then moves towards A till D and A coincide. The person at C makes the needed minor modifications in the position of C so that both the portions CA and CB are tight, producing a right angle at B. Now, these people did not worry why this produces a right angle. It just produced a right angle and they used it. They had collected a large source of such *tricks* which worked satisfactorily. Just as they say that the *proof of the pudding is in eating it**, so these ancient people preserved the rules that work; fulstop! However, the Greeks had a certain way of thinking. Before they used a rule, they would like to know *why* did it work. One reason for such an attitude appears to be that contrary to the popular belief "*Seeing is believing*", they had learnt *otherwise*. Let us appreciate their difficulty by the following experiments.

I. *Bending Lines*: Draw two parallel straight lines AB and PQ. In between these lines, take a point O(Figure 3). Through the point O, draw a large number of lines in all directions. Goodness me! What have you done to the lines AB and PQ? How did you manage to *bend* these lines around O? Naughty, that is what you are!

*That is, if you can *eat* the pudding, then *it* is there; you do not have to *show* by argument that the pudding exists.

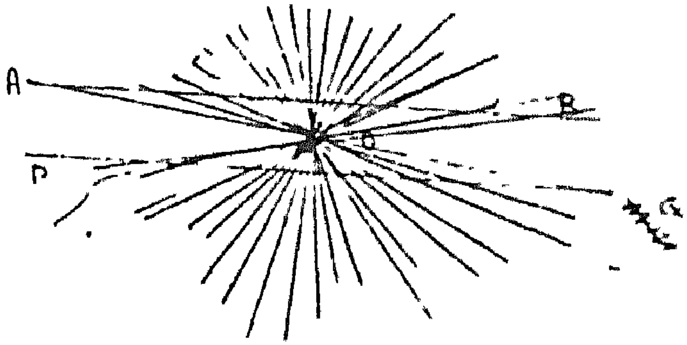


Figure 3

II. Equal but Unequal: Look at figure 4 which of the line segments AB and PQ is longer?

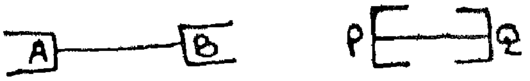


FIG.4

Look at figure 5. Which of AB and PQ is longer?

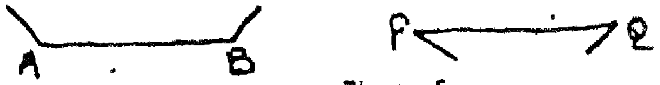


Figure 5

Look at Figure 6. Which of AB and PQ is longer?

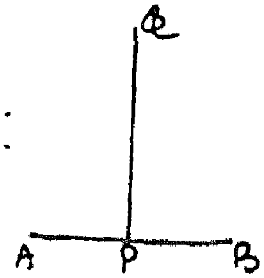


Figure 6

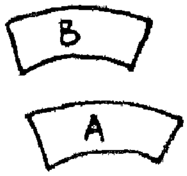


Figure 7

Look at Figure 7. which of the region A and B has bigger area? Now measure AB and PQ. In every instance you would find that AB = PQ. Also area A = area B. Is seeing knowing the truth?

III Always a Prime: Consider $n^2 - n + 41$. Give to n the values 1, 2, 3, 4, 5, ... Do you always get a prime number? So, shall we say that "$n^2 - n + 41$ is a prime number for all positive integers"? When n = 41,

we get

$$n^2 - n + 41 = 41^2 - 41 + 41 = 41^2$$

Alas! There goes down our card - fort! Upto $n = 40$, $n^2 - n + 41$ always produces a prime but not any more necessarily.

Such instances as above led the Greeks ~~by~~ [illegible] to believe anything only when they had proved it logically by faultless arguments on the assumption of certain self-evident truths and suppositions made. Geometry provided a very suitable model for their thinking. They assumed certain facts about points and lines and built a whole structure of results (theorems) on their basis. Their point had no dimension. In other words, it had neither length, breadth or height. Plot it please! Now do not show me a pencil dot. It had a length, a breadth and also a height, no matter how small. Have another try. Fold a piece of paper and press it to create a crease. Open the paper and as before form another crease, not parallel to the first one please ! These two creases cross each other at a point? Oh no! This too is not the point as defined by the Greeks ! This too has a length, breadth etc. though less than before! The point my dear is that their point is invisible. How can you see something which has no dimensions? So the dot is only a physical representation of point and helps in understanding the results about the point which greeks obtained. Similarly for other geometric concepts of the Greeks ! However, the beauty of Greek geometry is that whatever they discovered about their invisible or abstract points, lines, angles, triangles, quadrilaterals, circles, other plane geometric shapes, spheres, cones, cylinders, prisms and what not, turns out to be true for all ordinary practical purposes about the physical representations of points, lines, angles, etc. as we have come to look upon them. For this reason, the study of geometry becomes useful and important to us also.

There are other reasons too why we should be interested in geometry. All around us we find numerous geometric objects - some natural, some man-made. Anywhere, where there is a question of shape or form, it is matter within

the boundaries of geometry. Nature is full of geometric forms. The study of universe itself needs studying geometry. People first thought that Earth is flat. But then why should we lose a ship out of our sight when we are using a powerful telescope? So earth is not flat. People thought it was spherical. Again there were problems. That led to the theory of a somewhat pear-like Earth. The paths along which Earth and other planets move, have been found to be geometric shapes which Greeks had studied 2000 years ago as something abstract, invisible, only in thoughts! At a time, we thought that the universe is made up of atoms which are sphereical in shape. This is no longer supposed to be true. Instead, other shapes are being proposed. One popular view is that the universe is made up of endless strings(i.e. loops). These loops keep on moving, and thus colliding with others. This results in breaking and fusing and forming other strings, and so on.

In the man-made environment too, geometry plays a great role. All around us we build objects in geometric shapes. Raise your eyes and you would observe geometric shapes. The whole of architecture - the art of constructing buildings, bridges and so on is geometry out and out. No wonder architecture is described as frozen geometry. Not only today, we knew the art of building special structures for different purposes 3000 years ago. Sulva-sutras describe many constructions with the help of ropes. Astronomy was also at its peak, Surya Sidhanta being the most famous. Thus apart from being a training in logical thinking(as per Greeks), geometry is our cultural heritage too!

Having appreciated the practical utility of geometry, it would be fun to discover that geometry provides a very fertile land for having fun. I shall restrict myself to paper-folding even though there is a lot more to the story. You already know how to fold a line, and a point.

Folding a' Square: Take an ordinary rectangular sheet ABCD of paper(Figure 8) Pick up the end C and bring it to lie on AD(Figure 9). The edge DC of the

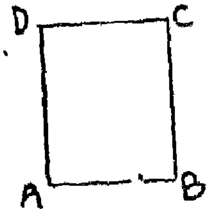


Figure 8

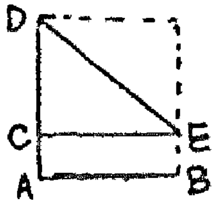


Figure 9

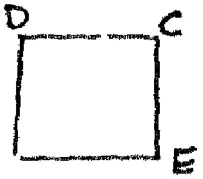


Figure 10

of the sheet becomes CE now. Fold back(or snip off) the portion ABEC along CE and move the corner C back to its original position(Fig. 10). What we get is a square. Can you justify this assertion? Incidentally, we have also folded an isosceles right-angled triangle. Which one? Can you prove this assertion?

Polygons: A polygon is a many-sided closed figure formed from straight line segments so that each segment is joined to two others at its extremities and no two of them are collinear or cross each other. Thus a triangle is a three-sided polygon and a quadrilateral is a four-sided polygon. A five-sided polygon is known as a pentagon(Figure 16), a six-sided polygon is called hexagon (Figure 17), and a seven-sided polygon goes by the name heptagon (Figure 18). What do you think are an octagon, a nanogon and a decagon? A polygon is called regular if all its sides and angles are equal. What is a 3-sided regular polygon? What is the name given to a 4-sided regular polygon? We shall now learn how to fold a regular pentagon.

Figure 16

Figure 17

Figure 18

Folding a regular pentagon : Take a strip ABCD of paper, 2 to 3 cm. wide and 20 to 25 long (Figure 19(a). Without crumpling the strip, try to form an ordinary knot by bringing over the right edge AB of

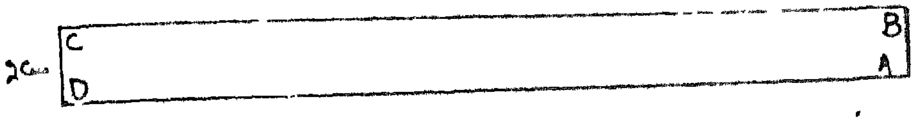


Figure 19(a)

the strip over the left edge CD and passing it through the loop so formed(Figure 19(b)). Notice that the letters A and B would not really be visible at this stage because they face towards the ground when

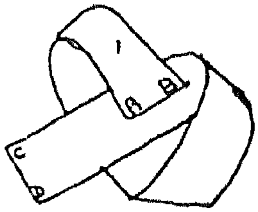


Figure 19(b)

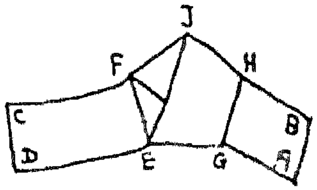


Figure 19(c)

the loop is formed. Now gentally pull out AB and smoothen out the knot (Figure 19(c)). Cut off the extra portions DEFC and AGHB along EF and GH respectively. You are now left with a regular pentagon EGHJF. Can you prove that the sides are all equal? If you hold this pentagon against light, you would observe a star known as the mystic pentagon. This was the emblem of the Society formed by Pythagoras to pursue the study of mathematics. Mathema is actually the Greek word for science.

You can cut out the pentagram as follows. Join JE, EH, HF, FG and GJ. Snip off the triangles on the sides of the pentagon(Figure 20). Try folding a regular hexagon and octagon.

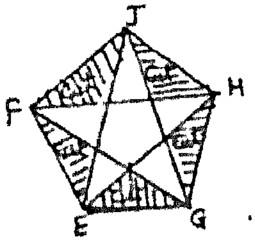


Figure 20

The figures folded so far were all figures with corners. We shall now fold a smooth figure(shape if you like !) having no corners.

Folding a parabola: Take an ordinary rectangular sheet of paper. Half way to the right, a point P roughly 3 cm above the bottom ege of the sheet(Figure 21). Now pick up the bottom edge and fold it slantwise so that it passes through P. Now press it to form a good crease(Figure 22). Open out and smoothen the paper. Fold again at a different angle taking care to make the bottom edge pass through P again. Press to form a crease. Open out and fold again at a different angle. Press for a crease. Repeat the process about 25 times. You would find that the creases shown dotted in figure 24 seem to enclose a curved shape.

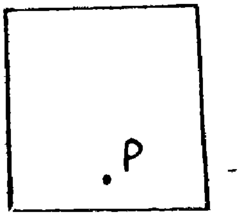


Figure 21

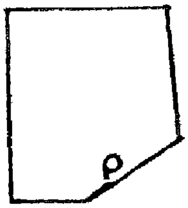


Figure 22

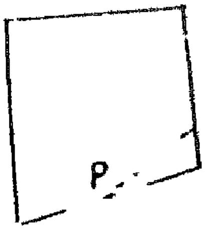


Figure 23

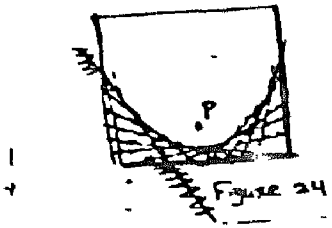


Figure 24

This shape is known as a parabola. When you throw a cricket ball, it traces a certain path in the air. The path is actually a part of some parabola. Similarly, a bullet fired from a gun traces portion of a parabola in the air. Parabola is a very interesting curve. It is used in the search lights. If we rotate the parabolic shape in figure 24 about a vertical line through P we shall get the surface ~~24 about~~ ~~vertical line through~~ of the shining shape that houses the bulb in a search light. If we move, this shape always keeping it parallel to its original position we shall get the surface of a reflector used in sun-ovens. All telescopes use a parabolic reflector. The vertical line through P is known as the *axis* of the parabola and P is known as the *focus*. When a ray of light parallel to the axis strikes the

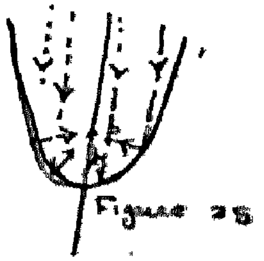


Figure 25

parabola, it is reflected back and always passes through the focus P. Thus a lot of light is collected at P and makes any object placed there better visible. Using this principle a Greek mathematician Archimedes around 250 BC was able to burn the enemy ships standing in the sea.

Threading a parabola: Another interesting way to obtain a parabola is through lines or threads. Take a piece of paper and draw an acute angle BOA with OA = OB. Mark fifteen points on each arm at equal intervals (Figure 25(a)). Number them as shown. Notice that the numbers go on increasing along OA but decrease along OB. Now join the points which have been numbered alike by means of straight line segments. There our friend parabola, sitting so neat and snug inside the angle BOA.

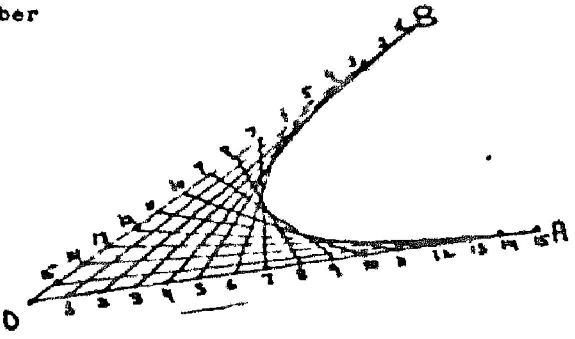


Figure 25(a)

Before proceeding further, make a parabola in angles measuring 30°, 45°, 60°, 90° and 120°. What do you notice? As you increase the angle, how does the shape of the parabola change? Now make the designs given in figures 25(b), 25(c), 25(d) and 25(e).

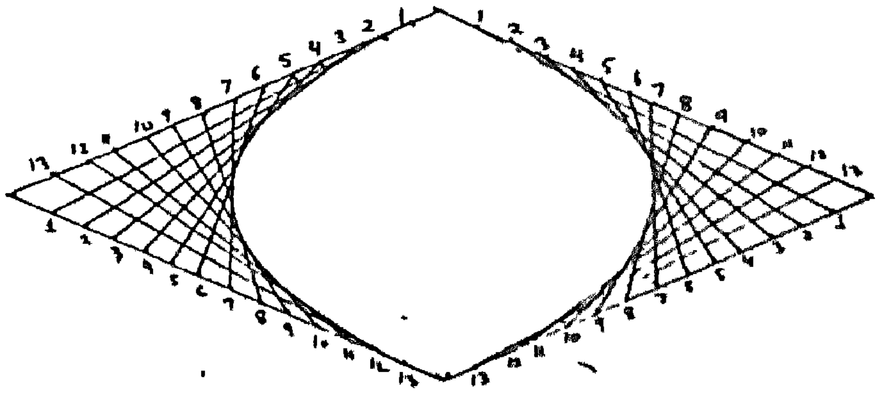


Figure 25(b)

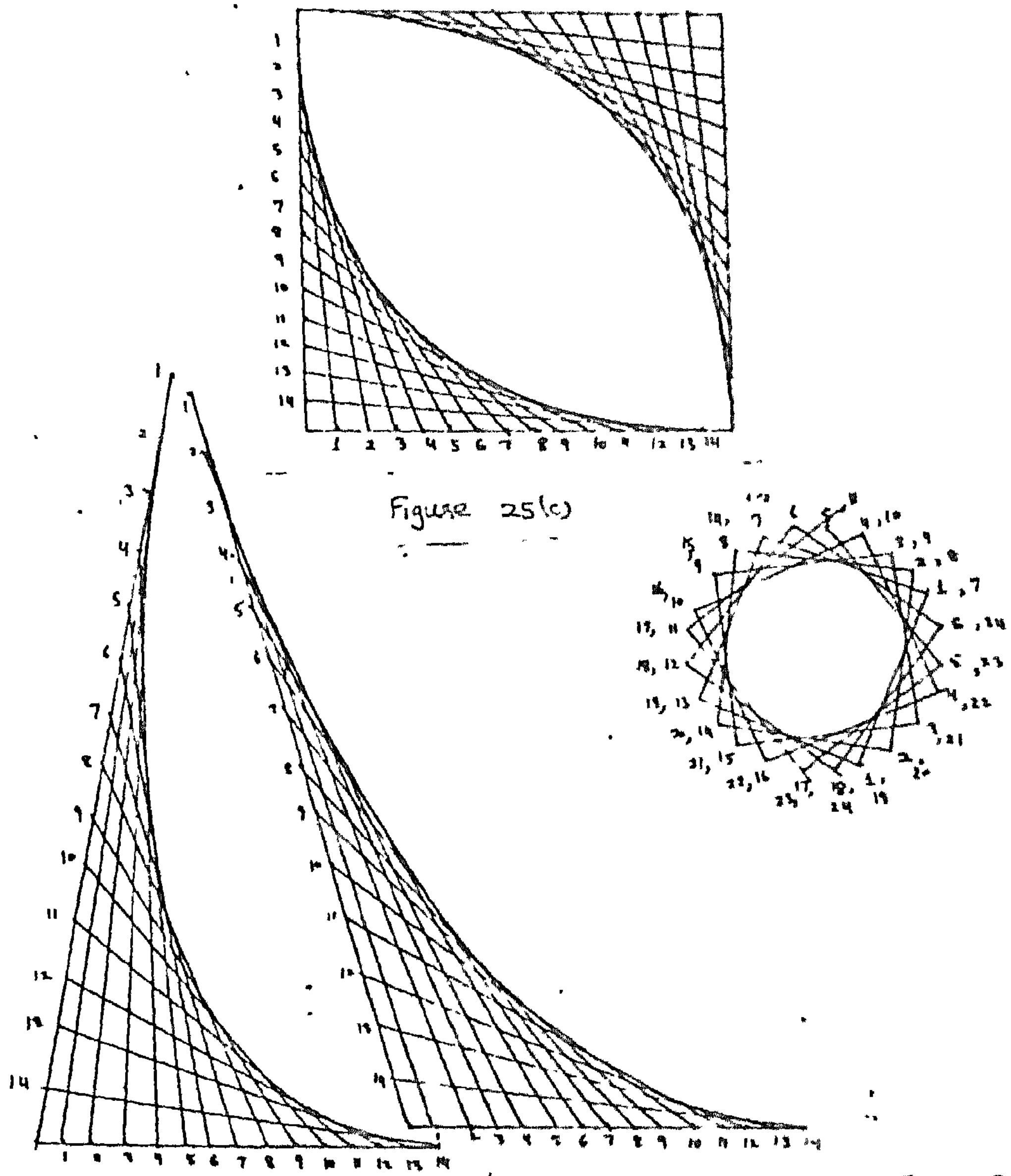


Figure 25(c)

Figure 25(d)

This time, the arms of the two angles have unequal length. Note carefully the number of points on eahh arm. Exactly the same ! This naturally makes the distance between the points on one arm different from that on the other. BUt points on the same arm are to be chosen at equal intervals.

Did you like your sail-boat?

* Circle not shown in the diagram.

You can make the sun shine too. Take a number of points on a circle* at equal distances. For this we take the tips of radii at equal angles. Here, the angle between two consecutive radii is 15°. Number the points 1 to 24 once. Then start numbering 1 to 24 once again but leave a few dots between the two 1's. Again join 1 to 1, 2 to 2 and so on.

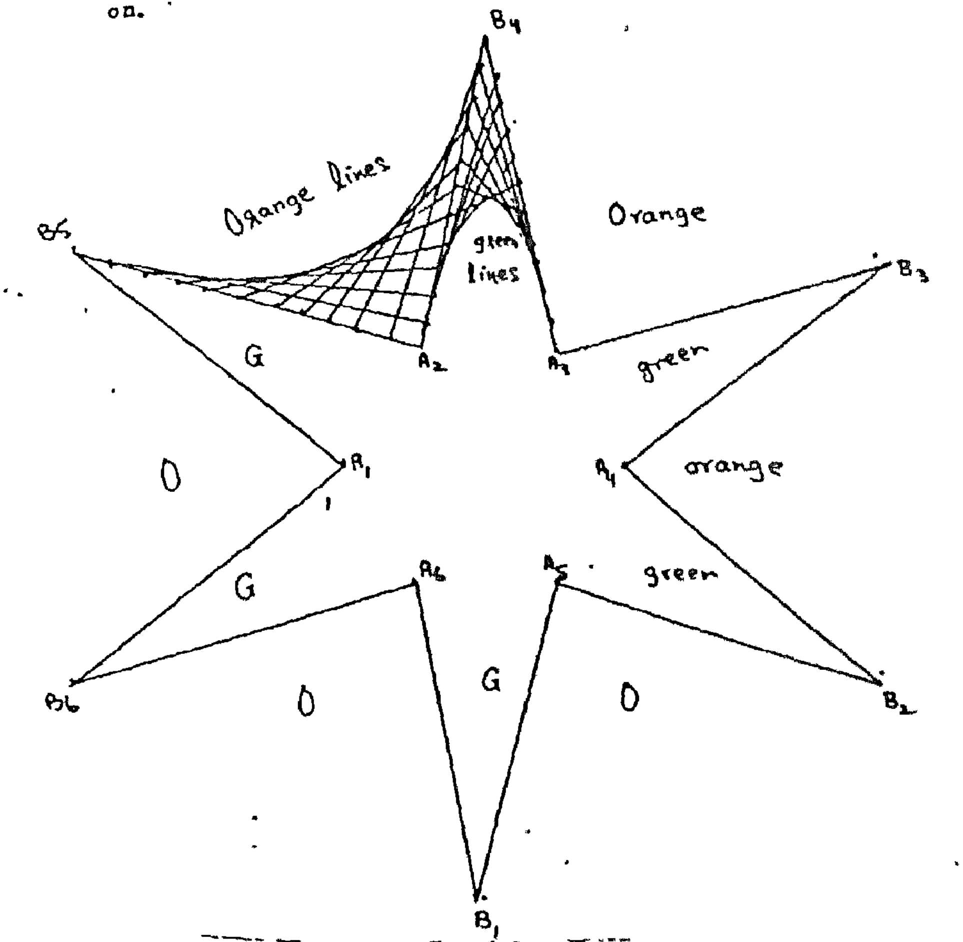


Figure 25(e)

The points A_1, A_2, ...A_6 lie on a circle at equal intervals. The circle is not shown. Similarly, the B points lie on a concentric cricle. Make the outer six parabolas in the angles at A_1, A_2,.....A_6 in orange lines. Make the six parabolas in the angles At B_1, ...B_6 in a different colour, say green.

Think of your own designs. Some of you may like to do these designs on stiff paper(or card sheet/chart paper), with a needle and coloured thread. In case you do, mark the points as before. But now prick a hole at the marks with the help of a needle. Start on the lower side of the card, coming up at point 1 on OA. Go down at point 1 on OB. Come up at 2 on OB and go down at 2 on OA. Come up at 3 on OA and go down at 3 on OB. Go on like this till all the marks are covered. Ultimately.

come up at the vertex of the angle and go over one arm. Come up at the vertex again and go over the other arm. Tape the end of the thread(and also the beginning) with scotch tape or quickfix etc. You can you use threads in various colours. The cards thus prepared can be used to decorate your room or classroom. You can also prepare greeting cards.

When one uses threads for making designs in this manner, the art is known as curve stitching or mathematical embroidery. Have you noticed designs in mathematical embroidery being sold in the market.

We shall now go back to our paper folding activity and see how we fold certain surfaces. This would involve some pasting also.

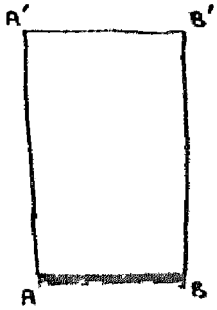


Figure 26

Folding a Cylinder: The simplest of all is a cylindrical tube. For this we need only a rectangular sheet AB B'A' of paper(Figure 26). To fold a cylinder, you paste the edges AB and A'B' with A going to A' and B going to B'. As a matter of fact, you have to overlap the sheet, the shaded portion going over the edge A'B' and then under the sheet. What would be the height of this cylinder shown in Figure 27? What would nearly be the circumference of its rim? What would be approximately the diameter of its circular end.

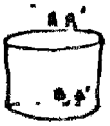


Figure 27

You can perform rather an interesting experiment with this tube. If not in open or near the window, stand(or sit) near a wall. Raise your left hand in front of your eyes, palm open and in front of the eyes at a distance of about 10 to 15 cm. Hold the tube in the right hand

with diameter roughly the length of your eye. Place the tube in front of your right eye, touching it or nearly so and place the left hand touching the tube.. Keep both eyes open and look far away or at opposite wall. Did something happen? Oh no ! A hole right through your palm ! and no pain ! Slide the hand a little bit this way and that along the tube Is the hole changing position on the palm? Don't ask me why this happens.

Folding a cube: A cube has six square faces. Cut out from a piece of card sheet or stiff paper a piece as shown in figure 28. Fold the sheet along the dotted lines one by one and press for a crease. Now raise the faces I, II, III and IV along the creases of the base to a vertical position. This should give you a hollow box-type thing with one face higher than the rest. Fold the protruding top face along the crease and manage it into a horizontal position parallel to the base. The flaps are to be used for overlapping and pasting. Once pasted, your cube is ready. You can paint it if you like. You can use it as a calendar for six months by writing (or pasting a piece from a calendar) the days and dates etc. You can write important results on the faces and keep it at a place where you would be looking frequently.

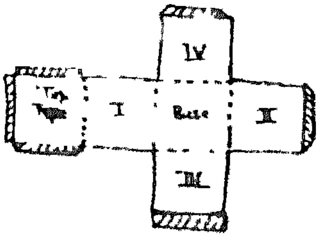


Figure 28

Folding a tetrahedron: A tetrahedron is a pyramid-like surface made up of four equilateral triangular faces having six edges and four vertices (figure 29). You can use the net in figure 30 to fold. Dotted lines are for creases and flaps for pasting are not shown. Make provision for them yourself. Actually flaps are not needed if you use scotch tape (cello tape). It is much easier to use the tape than to slip the flaps under or above and decide where to supply them.

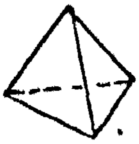


Figure 29

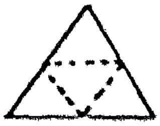


Figure 30

Can you make a tetrahedron out of six match sticks? Remember it has six edges!

Folding an Octahedron: An octahedron can be obtained by eight equilateral triangles, four triangles forming a pyramid on a square ABCD above the square, and the remaining four triangles forming a similar pyramid below the square(Figure 31). It has six vertices A, B, C, D, P and Q. How many edges does it have? How many faces does it have? Figure 32 shows a net for folding it. As before, dotted lines are to be creased and faces folded around these to the required position. Flaps for sticking are not shown. It is not necessary to have flaps on both sides. You can have a flap on one face, gum this flap and push it under the other of the two faces to be joined and stick.

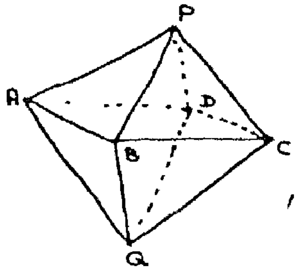


Figure 31

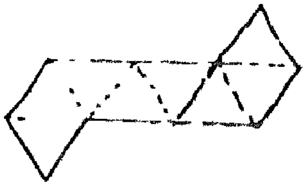


Figure 32

Folding a dodecahedron: Have you seen a dodecahedral calendar? I guess not ! How many faces does it have if it can be used as a calendar? Obviously, twelve! A dodecahedron has twelve pentagonal faces all congruent to each other. At each vertex three faces meet. Try to calculate the number of edges. Remember when you join two faces at an edge, they produce one edge of the dodecahedron we are folding. A dodecahedron and the required net are shown in figures 33 and 34 respectively.

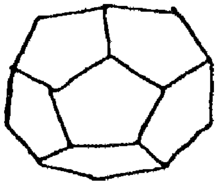


Figure 33

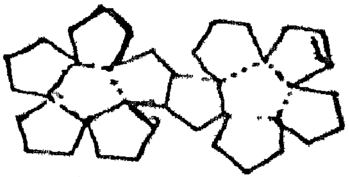


Figure 34

Folding an icosahedron: There is another surface known as icosahedron which completes a family consisting of cube, tetrahedron, octahedron and dodecahedron. It is made of twenty equilateral triangles, five triangles meeting at each vertex. Count the number of vertices and edges. The shape and net for it are given in figures 35 and 36 respectively.

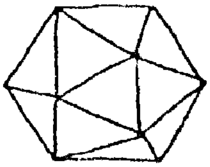


Figure 35

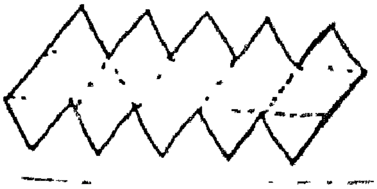


Figure 36

Notice that even though we have called the cube, tetrahedron etc. surfaces because we were concerned with the surfaces only, these bodies enclose volume and hence may be thought of as solids. Taken together the five above-mentioned bodies are known as the platonic solids. These are the only solids which can be made from regular polygons. Hence they are also known as regular polyhedra. The Greeks discovered the dodecahedron and the icosahedron. The other three had been known to the Egyptians.

I have demonstrated how you can fold some geometric plane figures and solids. There is a lot you can do in this area q which I have not mentioned. Try to find out what is origamy. With this task in hand I say good bye to you and hope and believe that you would have lots and lots of fun in the days to come, not by paper folding alone, but by learning the different types of geometry and by knowing how they have contributed to the development and growth of the scienfic thought. -

POSING PROBLEMS FOR MATHEMATICALLY GIFTED PUPILS

- SURJA KUMARI
D E S & M, NCERT,
NEW DELHI-110016.

For a long time, there has been general interest among researchers, curriculum designers and practising teachers in the area of problem-solving in Mathematics curriculum. This area gained momentum after the recommendations of the NCTM [1] in 'An Agenda for Action'. There is voluminous literature on various aspects of problem-solving but little attention has been paid to problem-posing which is as vital as solving it. Herein, we consider some important aspects of problem-posing for mathematically gifted students.

The word problem, in general as stated by Polya [2], is 'a situation in which a person searches consciously for some appropriate action to attain the clearly but not easily attainable aim. To solve the problem means to find such an action'. But one usually encounters different degree of difficulties to find out such an action and the problems are accordingly categorised as easy or difficult. In other words, some degree of difficulty belongs to the very notion of the problem.

In mathematics a problem is posed to reinforce the understanding of a concept or to use a skill learned earlier by presenting a situation that embodies that concept or skill. The problem posing for mathematically gifted students needs special attention as it is essential to present a situation which invoke their mathematical ability, interest

and imagination. Routine type problems generally lead to disinterest and dissatisfaction in the gifted pupils. Here, we shall try to develop such problems which help them to generate thinking potential, special ability and the maximum involvement on the part of the pupil and the minimum by the direction of the teacher. Following Krutetskii [3] model, the problems can be posed in the following categories:

I. Gradual Transformation from Concrete Situation to Abstract

Today, more than ever before in the history of human being, the importance of mathematics in the affairs of everyday living is being recognised. So the problems posed with concrete data, specially from children's immediate environment, help to reinforce the basic concepts of mathematics learnt in the classroom. For example, The length of a classroom is 10m, its breadth is 12m and its height is 3m. What is the volume of 2 such rooms?

Such problems posed with concrete data, can easily be transformed into abstract situation - the length of room is ℓm, its breadth is bm and height is h m. What is the volume of n such rooms?

This way the gifted students can be trained to express relations between quantities in a generalised abstract form.

II. Pupil's Perception of Mathematical Problem:

Here the question is stated neither directly nor indirectly but the student is asked to complete an incompletely stated problem. The student who picks up quickly to the logic of the basic relations and dependencies given in the problem would be able to complete the statement of the question and which reveals his/her mental perception of

of the mathematical problem. For example:

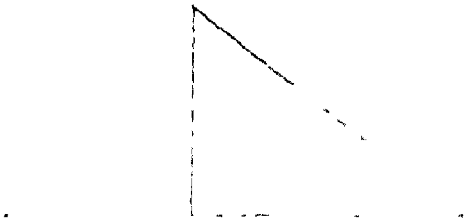
1. Anita bought 2b notebooks at one shop and 4.5 times as many at another. (How many notebooks did she buy all together).
2. On a straight line a point is given from which 2 rays are constructed on the same side of the line. The measure of one of the angles is given and the size of another is half the first angle. (Determine the size of the third angle).

The statement given in the parenthesis are to be completed by the pupil.

III. Problems with Verbal and Visual Formulation:

Recapitulation is important for any learning process. Problems reinforcing this skill substantiate better understanding of the concepts learnt in the classroom. For example, a problem like 'In the algebraic expression given below what are the exponents of each of the literals? Then asking for an appropriate definition of an algebraic expression and its degree
$2a^3$; b^3; $-3x^2y^2$; $7xy^6$; $\frac{1}{2}mn$; z; $C/4$ help to verbalise the idea of an algebraic expression.

If a problem is posed through pictorial or concrete means, pupils have an opportunity to use their ingenuity to perform geometric transformations of the objects or to visualise a physical representation of a problem. For an example, 'determine the validity of the statement that a perpendicular from a given point to a given line is shorter than any other slanting line segment from that point to the line'.



Such problems provide correlation between verbally expressed concepts and the graph.

IV. Composition of Equation:

Problems like 'A teacher told a student that she had added 12 to a given number and divided the result by 13. But the student was not paying close attention and substracted 13 from the given number and divided the result by 12. He gave the correct answer. What is the given number? help to generalise a definite method of reasoning. Moreover, such problems will develop pupils' habit of composing equations as well as definite general principle of approach to the solution of such mathematical problems.

V. Problems related to Special Concepts:

Problems involving the idea of special concept both in two dimensions and in three(inspace) will widen the imagination power of the gifted students. For example: Through a point outside a plane, how many straight lines can be drawn parallel to the plane? What solid is obtained by rotating a right triangle about an axis parallel to one of its legs?

VI. Problems on Comprehension and Logical Reasoning:

In this category the problems posed should reveal retention, reproduction and memorizing of mathematical material. For example:

Find the smallest number that gives a remainder 1 when divided by 3, a remainder of 2 when divided by 4, a remainder of 3 when devided by 5 and remainder of 4 when devided by 6.

The ability to solve such problems are a good indicator of the presence of mathematical abilities and logical reasoning power of the pupil.

VII. Problems on Proof:

The essential quality of mathematics which binds it together in a coherent way is the use of mathematical proof to deduce new results from known ones thus building up a strong and consistent theory. In writing a mathematical proof one needs a sense of judgement as to how much detail is really appropriate to the given task and what must be left out. Pupils are trained to acquire this skill through problem-solving situations. The problems like 'prove that the bisectors of the base angles of an isosceles triangle are equal' reinforce this skill in pupil.

VIII. Problems with several solutions:

Here such problems are posed which can be solved in different ways. The student has to find on his/her own the maximum number of methods of solving the problem. For example,
In how many ways can Rs.70 be paid if the money is in Rs.5 and Rs.10 notes only.

Prove in different ways that only one perpendicular can be dropped from a point outside a straight line(two methods). Such type of problems

are extremely useful as they inculcate the habit of creative thinking and applications of the concepts learnt.

IX. 'What if not' Method:

Lastly, we shall try to pose problems on 'what if not' technique developed some years ago by Marison [4]. The technique is to identify the attributed of a theorem, problem or puzzle and negate them one at a time. Each negation modified the original problem and offers the potential for generating new problems. For example, we consider the famous 1089 puzzle. Here we take a three digit number in the base ten such that digit at one's place is at least 2 more than at the hundred's place. Form a second number by interchanging the digits at one's and hundred's place. Find their difference. Interchange the digits in hundred's place and one's place again. Then add the numbers so formed. We see that the number is 1089.

For example:

(2)

$$\begin{array}{r} 604 \\ -\ 406 \\ \hline 198 \end{array} \qquad \begin{array}{r} 198 \\ +\ 891 \\ \hline 1089 \end{array}$$

$$\begin{array}{r} 833 \\ -\ 338 \\ \hline 495 \end{array} \qquad \begin{array}{r} 495 \\ +\ 594 \\ \hline 1089 \end{array}$$

Notice that in all the cases the resulting number is 1089. Why?

To apply the technique we first identify the attributes of the problem step by step and then solve it. The second stage of the technique is to negate attributes of the problem one at a time, and then based on such negation attempt to generate new solutions.

Lastly, we feel that if problems are posed under above said categories, the gifted students will definitely find mathematics interesting and application oriented subject and will help them to generate thinking potential at the school stage.

References:

1. National Council of Teachers' of Mathematics. An Agenda for Action: Recommendations for School Mathematics of the 1980s.
2. G. Polya, 'How to Solve it' 1957.
3. V.A. Krutetskii, 'The Psychology of Mathematical Abilities in School Children'. Edited by J. Kilpatrick, the University of Chicago Press.
4. W. Marison and S. Brown, 'What if not' Mathematics Teacher 46, 1969.

SAMPLE PROBLEMS FOR UPPER PRIMARY TEACHERS

Surja Kumari
D.E.S.M.
N.C.E.R.T.

1. Inscribed in a square with side a is another square, whose vertices lie on the sides of the former. Determine the segments into which the sides of the first square are divided by the vertices of the second square, if the area of the latter is equal to 25/49 of that of the former.

2. A triangle ABC is divided into three equal figures by straight lines which are parallel to the side. Compute the parts into which the side AB, equal to a is divided by the parallel lines.

3. Given inside a circle, whose radius is equal to 13 cm, is a point M 5 cm distant from the centre of the circle. A chord AB=25 cm is drawn through the point M. Find the length of the segments into which the chord AB is divided by the point M.

4. In a right-angled triangle a semicircle is inscribed so that its diameter lies on the hypotenuse and its centre divides the latter into two segments equal to 15 cm and 20 cm. Determine the length of the arc of the semicircle between the points at which the legs touch the semicircle.

5. Find the weight of an artillery round, knowing that the charge weighs 0.8 kg, the weight of the projectile is equal to 2/3 of the total weight of the round, and the weight of the shell is 1/4 of the weight of the round.

A dog at the point A goes in pursuit of a fox 30 metres away. The dog makes 2m and the fox, 1 m long leaps. If the dog makes two heaps to the fox's three, at what distance from A will the dog catch up with the fox.

Two bodies moving along the circumference of a circle in the same direction come together every 36 minutes. If they were moving with the same speeds as before, but in opposite directions, they would meet every 8 minutes. Also, when moving in opposite directions, the distance (along the circumference) between the approaching bodies decrease from 40 metres to 26 metres in 24 seconds. What is the speed of each body in metres per minute and how long is the circumference.

Is it possible for a farmer to plant ten trees so that there are five rows of four trees.

A man entered an orchard through 7 gates and picked some apples. When he left, he gave the first gaurd half his apples and 1 apple more. To the second gaurd he gave half his remaining apples and 1 more. He did the same to each of the remaining five guards and left the orchard with 1 apple. How many apples did he gather in all.

A 250 metre long oil tanker is cruising at 20 Km/hour. A small power boat travelling at constant speed takes one minute to go around the tanker, starting and finishing at the ship's stern. Find the power boat's speed. Ignore the width of the tanker.

Show that the product of five consecutive positive integers cannot be a square of an integer.

12. PQRS is a quadrilateral of area A. O is a point inside PQRS. Prove that if $2A=OP^2+OQ^2+OR^2+OS^2$, then PQRS is a square and O is its centre.

13. The binary expression for a certain multiple of 17 contains the digit "1" exactly three times.
Prove that it contains the digit "0" at least six times and that if it contains the digit "0" exactly seven times, then it is even.

14. ABC is a triangle. The internal bisector of the angle A meets the circumcircle again at P. Q and R are similarly defined with B and C respectively.
Show that
$AP+BQ+CR > AB+BC+CA$.

15. Find two digit numbers having the following property: if we adjoin zero followed by the smaller number on the right of the larger one, and adjoin the larger number followed by zero on the right of the smaller one, then of the two five-digit numbers thus obtained the first number divided by the second yields a quotient of 2 and a remainder of 590. It is also known that the sum of the two-fold larger desired number and the three-fold smaller desired number is equal to 72.

16. Two internal combustion engines of the same power output were subjected to an efficiency test and it was found that one of them consumed 600 grams of petrol, while the other, which was in operation 2 hours less, consumed 384 grams. If the first engine had consumed as much petrol per hour as the second,

and the second, as much as the first, then both engines would have consumed equal amount of petrol during the same period of operation as before. How much petrol does each engine consume per hour?

17. A sports ground has the shape of a rectangle with sides of a and b metres. It is bordered by a running - track whose outer rim is also a rectangle whose sides are parallel to and equally spaced from the sides of the ground. The area of the track is equal to that of the ground. Find the width of the track.

18. Ms. Farida puts money in a savings bank and one year later earns an interest of 15 rupees. Having added another 85 rupees she deposits the money for another year. After the expiry of this period the sum-total of the principal and its interest is 420 rupees. What sum of money was originally deposited and what interest does the savings bank pay?

19. The base of a triangle is divided by the altitude into two parts equal to 36 cm and 14 cm. A straight line drawn perpendicular to the base divides the area of the given triangle into two equal parts. Into what parts is the base of the triangle divided by this line?

20. In an isosceles triangle the length of the side is equal to a and the length of the line-segment, drawn from the vertex of the triangle to its base and dividing, the vertex angle in the ratio 1:2 is t. Find the area of the triangle.

21. A bear, starting from the point P, walked 1 km due south. Then he changed direction and walked one km due east. Then he turned again to the left and walked 1 km due north, and arrived exactly at the point P he started from. What was the colour of the bear.

22. Ramlal wants a piece of land, exactly level, which has four boundary lines. Two boundary lines run exactly north-south, the two others exactly east-west, and each boundary line measures exactly 100 metre. Can Ramlal buy such a piece of land in India?

23. Kasim has 10 pockets and 44 one rupee coins. He wants to put his coins so distributed that each pocket contains a different number of coins. Can he do so?

24. To number the pages of a bulky volume, the printer used 2989 digits. How many pages has the volume?

25. Among Grand fathers papers a bill was found:

72 hens Rs. - 67.9 -

The first and last digit of the number that obviously represented the total price of those hens were replaced by blanks, for they have faded and are now illegible. What are the two faded digits and what was the price of one hen?

26. Given a regular hexagon and a point in its plane. Draw a straight line through the given point that divides the given hexagon into two parts of equal area.

27. The vertex of a pyramid opposite the base is called the apex. (a) Let us call a pyramid 'isosceles' if its apex is at the same distance from all vertices of the base. Adopting this definition, show that the base of an isosceles pyramid is inscribed in a circle the centre of which is the foot of the pyramid's altitude.

28. The length of the perimeter of a right triangle is 60 inches and length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides.

29. Consider the table

$$1 = 1$$
$$3+5 = 8$$
$$7+9+11 = 27$$
$$13+15+17+19 = 64$$
$$21+23+25+27+29 = 125$$

Guess the general law suggested by these examples express it in suitable mathematical notation and verify it.

30. The side of a regular hexagon is of length n (n is an integer). By equidistant parallels to its sides the hexagon is divided into T equilateral triangles each of which has sides of length 1. Let V denote the number of vertices appearing in this division, and L the number of boundary lines of length 1. When n=1 find T V and L. Next consider the general case and express T,V and L in terms of n.

ENRICHMENT TRAINING FOR RESOURCE PERSONS IN MATHEMATICS AT UPPER PRIMARY STAGE

Venue: National Institute of Education
NCERT, New Delhi
4th September to 8th September, 1988

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Pin Code – 152 101 (Pb).

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N.V. Pabra
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4. Sh. Kashmiri Lal Kashyap
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5. Ms. Shashi Jain
Kendriya Vidyalaya No. 3,
Ambala Cantt.

6. Ms. M. Misra,
Primary Teacher
Blue Bells School,
Kailash,
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7. Ms. Gurpreet Kaur
Navodaya Vidyalaya
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8. Ms. Sushila Kumar
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Central School For Tibetans
Darjeeling (W) 734 101.

16. Sh. P. Subbaraya Sarma
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Andhra Pradesh.

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Kendriya Vidyalaya Army No. 1,
Jodhpur, RAJASTHAN

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34/2", Old Rajinder Nagar,
New Delhi 110060.

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Appejay School,
Panch Sheel Park,
New Delhi.

20. Prof. J.N. Kapoor,
C 766, New Friends Colony,
New Delhi 110065.

21. Prof. Asha Rani Singal
Mathematics Department
Meerut University
Meerut.

22. Dr. N. Puri
Civil. Eng. Department
Roorkee University
Roorkee.

23. Dr. S.P. Arya
Department of Mathematics
Maitreyi College,
Bapudham Complex
Chanakya puri
New Delhi 110021.

24. Dr. Sunder Lal
Reader
Department of Mathematics
Agra University
Agra.

N.C.E.R.T. Faculty Members.

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